Dynamic Optimal Control Models in Advertising: Recent Developments

Gustav Feichtinger • Richard F. Hartl • Suresh P. Sethi
Institute for Econometrics, Operations Research and Systems Theory, Technical University, Vienna, Austria
Institute for Econometrics, Operations Research and Systems Theory, Technical University, Vienna, Austria
Faculty of Management, University of Toronto, Toronto, Ontario, Canada M5S 1A4

This paper presents a review of recent developments that have taken place in the area of dynamic optimal control models in advertising subsequent to the comprehensive survey of the literature by Sethi in 1977. The basic problem underlying these models is that of determining optimal advertising expenditures and possibly other variables of interest over time for a firm or a group of competing firms under consideration. This optimization is done subject to some dynamics that define how these variables translate into sales and in turn, into profits. The purpose of this update is twofold. On the one hand, new contributions in the areas already treated in the earlier survey are reviewed. On the other hand, new trends in the advertising literature since 1977, such as quality as an additional marketing instrument, cumulative sales models, pulsing advertising, and advertising as a part of corporate models of the firm, are discussed. The models covered in this update are organized under six headings: models with capital stocks generated by advertising, price, and quality, sales-advertising response models, cumulative sales models for durable goods, models with more than one state variables in the advertising process, models incorporating interaction with other functional areas, and competitive models. The discussion involves specifications, methods used, results, empirical validation, if any, and their economic significance. The survey concludes with suggestions for extensions and future directions of research.

(*Advertising Models; Optimal Control Theory, Maximum Principle; Market Growth Models; Competitive Models; Pulsing*)

Contents

0025-1909/94/4002/0195$01.25

List of Symbols

*	desired or optimal levels
‾	singular or equilibrium levels
A	stock of advertising capital (goodwill)
$\bar{A}$	adaptation level
B	advertising budget
$C(u)$	cost of advertising
C_p, C_I, C_v	cost coefficients in decentralized marketing-production planning
$E(p, A, Z)$	rate of profit margin gross of advertising
F	cumulative distribution function of market entry
H	current-value Hamiltonian
I	inventory level
I_i	investment in the copy design for the advertising campaign number i
L	current-value Lagrangean
M	saturation level of sales rate
$\mathbf{M} = (m_{ij})$	quadratic matrix of response coefficients
N	market size
P_i	probability of i units of goodwill
$P_j(I_i)$	probability that the initial performance of an ad is $\xi_j(I_i)$
Q	expected quality
R	reputation
S	sales (demand)
$\bar{S}$	salvage value
T	planning horizon
T_i	copy replacement time
$U_i(u_i)$	advertising expenditures corresponding to u_i
V	variance in goodwill
W	standard Wiener process
X	cumulative sales
$\bar{X}$	market potential
Z	exogenous variables or factors; transfer variable
a	constant first purchase rate
b	repeat purchase rate
b_1, b_2	salvage values
c	unit cost of production
g	effectiveness of advertising
h	'experience function' describing the effect of the quality-price ratio upon R; hazard rate factor; effect of price on the dynamic demand function
$\mathbf{h}$	vector of weights
k	constant contact rate
l	survivor function
m_{ij}	coefficients measuring the influence of the price for item j on the sales of i
p	price per unit
$\mathbf{p}$	vector of actual prices
p^0	reference price; hypothetical price corresponding to a given quality
$\mathbf{p}^0$	vector of reference prices
$\bar{\mathbf{p}}$	vector of backstop prices
q	quality
r	discount rate
$\mathbf{s}$	vector of sales rates
t	time
u	rate of advertising
$\bar{u}$	maximum advertising rate
u_1	awareness advertising rate
u_2	trial advertising rate
u^r	rival's rate of advertising
v	number of shoppers visiting the store per unit time; rate of production
w	weighting function; differential stimulus effect
x	fraction of market potential captured or sales rate
x_1	number of people unaware of the product
x_2	number of potential customers
x_3	number of current customers
y	awareness
$\bar{y}$	threshold level for awareness
z	filtered advertising
Ψ	replenishment function
$\Pi(A, Z)$, $\Pi(A, q)$, Π_i	rate of profit margin gross of advertising when the price is optimal
α	decay rate of sales
β	elasticity of demand with respect to advertising; fraction reflecting all other variable costs
γ	advertising effectiveness coefficient; decay rate of reputation; relaxation parameter
γ_i	intensity of a Poisson process
δ, $\delta(q)$	rate of depreciation of goodwill or market share; rate of deterioration for inventories
ϵ	elasticity of the dynamic demand with respect to cumulative sales; small parameter associated with machine breakdown and repair rates
η	elasticity of demand with respect to price
$\phi(R)$	reputation response function
$1 - \theta$	rate of depreciation of goodwill (in a discrete time setting)
λ	shadow price (adjoint variable); birth rate
μ	transfer price; death rate
π	gross margin per unit (net of costs other than advertising)
ρ	response constant
σ^2	variance
ξ	initial performance of an advertising campaign
ζ	speed of adjustment
ω	elasticity of demand w.r.t. to goodwill
$\mathbf{E}$	expectation operator

1. Introduction and the Classification Framework

Sethi (1977a) wrote a comprehensive survey of the literature on dynamic optimal control models in advertising devoted to determining optimal advertising expenditures over time subject to some dynamics that define how advertising expenditures translate into sales and in turn, into profits for a firm or a group of firms under consideration. Fifteen years have elapsed since. In these intervening years, the control theory models, as can be seen from the list of references, have firmly established themselves in the marketing literature, whereas prior to Sethi's survey, they were viewed as newcomers to the literature.

Among Sethi's major conclusions were a dearth of competitive models reflecting the game aspects between the firms in the relevant markets and of models that incorporate the uncertain nature of advertising effects. Moreover a majority of the models were simple in order to permit a characterization of their optimal solutions. All of the models surveyed had optimal advertising dependent ultimately on time or in the case of stochastic models, on the cumulative information available by that time. Finally, Sethi concluded that while it is the most important part of a dynamic advertising model, the empirical work to discover sales-advertising relationships is far from complete.

Looking at the present state of the literature on dynamic optimization models in advertising, we see that some of the voids in the literature pointed out by Sethi have partially been filled. A large number of differential game models in advertising have appeared since then. Some progress has also been made in connection with the development of stochastic optimal control models in advertising. A few papers dealing with larger size models and computing their optimal solutions numerically have also been written. There also exists a paper by Seidman et al. (1987) using a distributed parameter framework allowing the optimal advertising expenditure to depend on such other dimensions as income levels, age, etc. in addition to time. On the other hand, the void that existed in the area of empirical tests of the models by and large continues to exist to this day.

In addition, there have been several new directions in which research has taken place. Some of these are indicated below.

Sethi (1977a) reviewed only one working paper, now published as Kotowitz and Mathewson (1979a, b), that introduced product quality as an important consideration. Incorporation of quality in addition to advertising and price has become a feature of many papers published in the last ten years.

Another important trend has been to develop optimal control formulations to determine advertising expenditures for durable goods. Dynamics in these problems involve diffusion of innovation (see Bass (1969)), product life cycle, cost learning phenomenon, etc.

There has also been a great deal of interest in developing models in which optimal advertising policies are pulsing policies. This literature is inspired by the classical paper of Sasieni (1971), in which optimal advertising policy is a "chattering" policy (i.e., policy of switching from one advertising level to another infinitely fast), and the critical review paper by Little (1979), who argued that chattering might not be perceived by consumers as pulsing.

Finally, there has been a trend toward integrating marketing models into corporate models. In these models, marketing must compete for corporate resources with other functional areas. Models incorporating these problems have been solved by using decentralized hierarchical control approaches.

From the above discussion, it seems appropriate that it is time to do an updated review of the control theory models in advertising. It is to this purpose that this paper will be devoted. What we have in mind is to review papers published subsequent to Sethi's survey, i.e., post-1977 papers on optimal control models in advertising. Occasionally, we shall describe some older papers for reference purposes, so that the paper can be read without too much reliance on Sethi (1977a).

We intend to write this paper for researchers in management science, marketing science, operations research, and optimal control. Our survey will not be too technical. However, the reader is expected to have some knowledge in differential equations and optimal control. For those unfamiliar with optimal control, reading a short tutorial on the topic by Sethi and Thompson (1981b) might be advisable before reading our survey.

1.1. The Classification Framework

The survey by Sethi (1977a) was organized in four model categories: advertising capital models, sales-advertising response models, micro models, and control-theoretic empirical studies. Advertising capital models considered advertising as an investment in the stock of *goodwill* as in the model of Nerlove and Arrow (1962). Sales-advertising response models were characterized by a direct relation between the rate of change in sales and advertising and represented various generalizations of the model due to Vidale and Wolfe (1957). The models grouped under the heading of micro models emphasized the specific mechanisms leading to their dynamics. Finally, empirical studies which used actual data to test various theoretical issues arising out of the optimal control models in advertising were described as control-theoretic empirical studies. With the new additional trends in the literature described above in the introduction, these four categories are no longer sufficient. What is called for is a new, broader classification, which we shall develop next.

The first category of Sethi (1977a) will be renamed as *capital stocks generated by advertising, price, and quality* to recognize the contributions of the additional control variables, price and quality, to the formation of advertising capital. The second category will stay as *sales-advertising response models*. The third category is a new category called *cumulative sales or market growth models*. This will include durable goods models, whose optimal control formulations are of more recent origin than 1977. The fourth category termed *models with more than one state variable in the advertising process* will replace Sethi's micro models. Here we shall include not only the models where consumers go through a set of stages leading from their exposure to a product towards a purchase of the product, but also the models that introduce additional variables in the advertising process to explain the optimality of pulsing advertising policies. Finally, our fifth and sixth categories are new categories called *interaction with other functional areas* and *competitive models* to recognize the new research trends in the literature pointed out briefly in the introduction.

We shall group post-1977 research in dynamic optimal control models in advertising in the six categories described above. Since there is a large number of papers to be surveyed and coverage of these papers may appear to be uneven to some, it may be useful to state the following remark pertaining to our style of coverage. In each of the sections where appropriate, we have first selected one particular model, which is formulated in some detail. It becomes convenient then to discuss other models in the section with reference to the selected model. While the criteria for our selection may be more or less arbitrary as far as the reader is concerned, the resulting discussion makes the paper concise and interesting to read at the same time. The alternatives would be to describe the formulation of important models for purposes of discussion thus increasing the length of the paper or to omit formulations altogether making it awkward to read.

The plan of the survey is as follows. Section 2 deals with models where capital stocks such as goodwill are generated by advertising, price and/or quality. In §3, we review sales-advertising response models. In §4, cumulative sales or market growth models for new durable goods are reviewed. Section 5 treats models with more than one state variable in the advertising process. One main topic is the question whether cyclical advertising or pulsing patterns can be optimal. In §6, we report on decentralized marketing models which describe interactions of advertising with other functional areas. Section 7 contains a brief survey of competitive models in advertising. Tabular summaries of results are provided for product quality models described in §§2.5 and 3.4, market growth models in §4, and pulsing models in §5.2. Each section is concluded by some general remarks which contain an evaluation of the models reviewed in that section, directions for further research or other additional remarks. Section 8 concludes the paper with suggestions for further research. Finally, we provide an extensive bibliography of post-1977 papers, although we may not cite all of these papers in the text.

Before proceeding to the next section, we should note that one could also classify models surveyed along such key dimensions as deterministic vs. stochastic, monopolistic vs. competitive, etc. That may cut across the categories suggested above. All of the models surveyed in this paper are deterministic except those in §§2.4, 3.2, 3.3, 6.3, and 7.1, which are stochastic. Only models in §7 are competitive; all remaining models are monopolistic. All of the models reviewed are profit-maximizing models. Quality is incorporated only in models in §§2.5 and 3.4. With respect to the dynamics, models in §2 represent generalizations of the Nerlove-Arrow dynamics, those in §3 represent generalizations of Vidale-

Wolfe dynamics, while those in §4 are inspired by Bass's market growth dynamics.

2. Capital Stocks Generated by Advertising, Price, and Quality

In this section, we review the post-1977 models, in which capital stocks such as goodwill or reputation are generated by advertising and (possibly) price and quality. These models typically are extensions and/or modifications of the classical model due to Nerlove and Arrow (1962) (NA, hereafter). The NA model is briefly described in §2.1 for reference purposes. Extensions of the model to allow for a limited advertising budget are discussed in §2.2. In §2.3, we discuss extensions of the NA model to incorporate lagged effects of advertising in the formation of goodwill. In §2.4, a stochastic extension in which goodwill is a random variable is discussed. Finally, we reserve §2.5 to review models that incorporate product price and quality as additional instruments in the optimal marketing mix. We note that incorporation of these decision variables has been a major recent trend in modeling.

2.1. Advertising Capital Models

An early seminal dynamic advertising model is that of Nerlove and Arrow (1962), who consider a *stock of advertising goodwill*, $A(t)$, which summarizes the effects of current and past advertising expenditures by a firm on the demand for its products. The advertising capital changes over time according to

$$\dot{A} = u - \delta A, \quad A(0) = A_0, \qquad (2.1)$$

where $u = u(t)$ denotes the current advertising rate (in dollars) and δ is a constant proportional depreciation rate. Every dollar of advertising outlay raises goodwill by one unit. The objective of the monopolistic firm is to maximize the present value of net revenue stream discounted at a fixed interest rate r, i.e.,

$$\max_{u\geq 0, p\geq 0} \int_0^\infty e^{-rt}[E(p, A, Z) - u]dt, \qquad (2.2)$$

where p denotes the price, Z summarizes all other variables not under the control of the firm, and $E(p, A, Z)$ is the total revenue net of production costs. Since the price does not enter the system dynamics (2.1), the revenue $E(p, A, Z)$ in (2.2) can be replaced by

$$\Pi(A, Z) = E(p(A, Z), A, Z) = \max_{p\geq 0} E(p, A, Z).$$

It should be clear from the above that the optimal price is not determined independently of advertising.

The optimal advertising policy in this problem is characterized by a most rapid approach to a turnpike goodwill level $\hat{A}$; see Spence and Starrett (1975), Sethi (1977b), and Hartl and Feichtinger (1987). This means that for $A_0 < \hat{A}$ it is optimal to jump instantaneously to $\hat{A}$ by applying an impulsive control at $t = 0$ and then set $u^*(t) = \hat{u} = \delta\hat{A}$ for $t > 0$; if $A_0 > \hat{A}$ the optimal advertising rate is $u^*(t) = 0$ until the stock of goodwill depreciates to the level $\hat{A}$, at which time the advertising rate switches to $u^*(t) = \delta\hat{A}$. Such a policy, it should be noted, is known as a bang-bang policy in the control theory literature; see, e.g., Sethi and Thompson (1981a).

An important result for the NA model is the dynamic counterpart of a theorem by Dorfman and Steiner (1954) saying that (in the long run) the advertising expenditures should be proportional to sales. The result offers support for the practice by some companies of setting this period's advertising expenditure in proportion to the previous period's sales.

Several nonlinear and other extensions of the NA model have been proposed. These, with the exception of the recent papers by Welam (1982) and Rao (1985), are reviewed in Sethi (1977a, §2). Welam's paper is a discrete time version of the model by Gould (1970), which has already been discussed in Sethi (1977a). The relationship between the models of Welam and Gould was pointed out by Rao (1985). Moreover, Rao also criticized the suboptimality of Welam's solution.

2.2. Limited Advertising Budget

Sethi (1977c) formulates an optimal advertising problem using the NA dynamics and linear advertising cost, that takes into account a limited advertising budget. Once again, a turnpike type solution is obtained. The level of the turnpike, however, may depend on the amount of the budget. Sethi and Lee (1981) extend the problem by considering a budget, which may be replenished over time by an amount depending on the level of goodwill. Finally, Hartl (1982) extends the analysis of Sethi (1977c) to nonlinear advertising costs.

2.3. Lag Models and Their Estimation

The NA dynamics assumes that there is no time lag between the advertising expenditures and the increase in the stock of goodwill. It assumes, however, that goodwill depreciates at a constant rate δ, which is equivalent to the assumption that the lifetime of each unit of goodwill is exponentially distributed with mean $1/\delta$. Nevertheless, the assumption of depreciation is not rich enough to include a variety of possible lag structures. Attempts have, therefore, been made to incorporate different distributions on the lifetime of each unit of goodwill into the dynamics of advertising capital.

A first class of such models is described by

$$A(t) = \int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau, \quad \int_{0}^{\infty} w(\tau)d\tau = 1, \tag{2.3}$$

where w is a weighting or density function. This means that the current goodwill $A(t)$ is a weighted average of past advertising expenditures u. Several important references using this approach are already surveyed in Sethi (1977a, §2.6); see also Tapiero and Farley (1975) for an extension to the case of multiple products and multiple salesmen.

A general formulation for the lag structure will combine two different lag structures: First, there is a time lag between the advertising expenditure u and the corresponding increase of goodwill. With $w(\cdot)$ denoting the density function of this time lag, the increase of goodwill at time t, $G(t)$, is

$$G(t) = \int_{-\infty}^{t} w(t-s)g(u(s), A(s))ds, \tag{2.4}$$

where $g(u, A)$ is the advertising effectiveness function (a production function for goodwill).

Second, there is a distribution of the forgetting time. By $l(\cdot)$ we denote the survivor function of the life time of each unit of goodwill. Thus

$$A(t) = \int_{-\infty}^{t} l(t-\tau)G(\tau)d\tau. \tag{2.5}$$

Combining Equations (2.4) and (2.5) one obtains

$$A(t) = \int_{-\infty}^{t} l(t-\tau)\int_{-\infty}^{t} w(\tau-s)g(u(s), A(s))dsd\tau \tag{2.6}$$

which can be rewritten as

$$A(t) = \int_{-\infty}^{t} g(u(s), A(s))\tilde{l}(t-s)ds \quad \text{where} \tag{2.7}$$

$$\tilde{l}(t) = \int_{-\infty}^{t} l(t-\tau)w(\tau)d\tau \tag{2.8}$$

fulfills all the requirements of a survivor function, i.e., $\tilde{l}(-\infty) = 0$, $\tilde{l}(\infty) = 1$, $\tilde{l}'(t) \geq 0$. This means that by combining the two lag structures (2.4) and (2.5) one can reduce the system dynamics (2.6) to a model with just one lag structure simply by redefining the survivor function according to (2.8).

Note that the NA model is the special case with $g(u, A) = 0$, $l(t) = e^{-\delta t}$ (i.e., the lifetime is exponentially distributed) and where $w(t)$ has its entire mass of 1 at $t = 0$ (i.e., there is no time lag between advertising and the increase of goodwill).

Pauwels (1977) considers the special case of (2.6) with $l(t) = e^{-\delta t}$ and $g(u, A) = g(u)$. By differentiating (2.6) in this case the system dynamics can also be rewritten as an integro-differential equation

$$\dot{A}(t) = \int_{-\infty}^{t} w(t-\tau)g(u(\tau))d\tau - \delta A(t).$$

Pauwels presents transient and steady-state properties of the optimal solution. In the special case of a gamma distribution, the sensitivity of the steady state w.r.t. the parameters is also investigated. In particular, a postponement of the peak effect of advertising lowers the long-run equilibrium rate of advertising, which is what one would expect.

Hartl (1984), on the other hand, has considered the special case of (2.6) where $w(t)$ has its entire mass of 1 at $t = 0$, i.e., where $G(\tau)$ in (2.5) is replaced by $g(u(\tau), A(\tau))$. It should be noted, however, that because of transformations (2.7) and (2.8), this formulation also captures the general case (2.6). For a general model with concave profit and production function, the optimality conditions are derived and the sensitivity of the steady-state solution is investigated. In particular, higher discounting rates imply smaller advertising expenditures and a smaller stock of goodwill in the long run. In a special case with $g(u, A) = g(u)$, $g'' < 0$ and profit function $\pi A - C(u)$, $C'' > 0$, π a positive constant, the transient behavior of the optimal advertising policy is also investigated. For instance, if the lifetime of goodwill is finite, i.e., $l(t) = 0$ for $t \geq K$, for some $K \in (0, \infty)$, then there is an initial interval where the optimal advertising expenditures should be constant and equal to their equilibrium value. In the case of a distribution characterized by an increasing hazard rate, $\dot{\delta} > 0$, a bell-shaped advertising trajectory is optimal, i.e., u increases on an initial interval, while it decreases in a second interval.

Along with the above theoretical developments, a number of applications of the econometric methodology have attempted to measure the lagged effect of advertising on sales over time. For this purpose, a discrete-time model with geometrically decaying lagged effects, i.e.,

$$A_t = \sum_{\tau=0}^{\infty} \theta^{\tau} u_{\tau}, \quad 0 < \theta < 1,$$

is considered and the well-known Koyck transformation is used to obtain a discrete version of the NA dynamics (2.1):

$$\Delta A_t = A_t - A_{t-1} = u_t - (1 - \theta) A_{t-1};$$

see also (2.3). Several variants of this simple but in some situations unrealistic geometric decline have been proposed, making it possible for advertising to reach its peak some time after the campaign is over; see, e.g., Montgomery and Silk (1972), Lambin (1976), Bass and Clarke (1972), Tsurumi and Tsurumi (1973), and Ward (1976).

Bultez and Naert (1979, 1988) and Magat et al. (1986, 1988) have investigated the effects of lag-structure in sales-advertising dynamics on the optimal advertising policies and profits of the firm. Based on the Lydia Pinkham data (see Palda 1964), Bultez and Naert come to the tentative conclusion that *"the current tendency to build and estimate increasingly sophisticated lag models does not seem totally justified."* Magat et al. explain their results with some specific properties of the Lydia Pinkham data so that no general conclusions can be drawn; see also Caines et al. (1977).

Rao (1986) estimates several advertising-sales models based on the following stochastic differential equation for goodwill A:

$$dA = [u - \delta A]dt + \sigma_2 dW,$$

where W is a standard Wiener process. Goodwill $A(t)$ is related to sales $S(t)$ by

$$S(t) = \beta_0 + \beta_1 A(t) + z(t),$$

where β_0 and β_1 are constants and z is a stochastic process affecting sales and modelled by $dz = -\rho z dt + \sigma_1 dW$. Clearly, the continuous variables cannot be observed, since observations on sales always correspond to temporally aggregated data. By imposing appropriate assumptions regarding spending over the data interval, Rao (1986) reduces his general model to the Blattberg and Jeuland (1981) model (see also Sethi 1983b). One of his conclusions is that under systematic advertising spending within an interval both models will yield biased estimates.

2.4. Random Walk Models

These models extend the NA model with the assumption of the uncertain nature of advertising effects. In these extensions, additions to goodwill and the depreciation of it are probabilistic effects. Specifically, it is assumed that in a small time interval, the probability that goodwill will increase by one unit is a function of the advertising rate. Similarly, in the same time interval, the probability that goodwill will decrease by one unit is a function of the forgetting rate. Finally, goodwill in the same time interval remains the same with the remaining probability.

Most of the random walk models including those discussed in Sethi (1977a) were developed by Tapiero (1978, 1979, 1988); see also Sethi (1979b). The construction of the probability $P_i(t)$ that goodwill (assumed to be discrete) is i at time t is described by Kolmogorov forward equations. Under some simplifying assumptions, it is possible to obtain the state equations representing the evolution of the mean and the variance of i over time. These can then be used in the formulation of a deterministic optimal control problem, which can finally be solved by using the maximum principle. These model types were reviewed earlier in some detail by Sethi (1977a) and will not be discussed here.

In this survey, we shall describe a simple model by Rishel (1985), who is able to solve it explicitly. Rishel assumes that $P_i(t)$, the probability that there are i units of goodwill at time t, satisfies the birth-death equations

$$\dot{P}_0 = -\lambda u P_0 + \mu P_1,$$

$$\dot{P}_i = \lambda u P_{i-1} - (\lambda u + i\mu) P_i + (i+1)\mu P_{i+1}, \quad i \geq 1,$$

with obvious interpretations for the parameters λ and μ and with the initial conditions $P_0(0) = 1$, $P_i(0) = 0$, $i \geq 1$.

The objective functional is

$$\max_{0 \le u \le \bar{u}} \int_0^\infty e^{-rt}\left[\pi \sum_{i=0}^{\infty} \gamma_i P_i(t) - u(t)\right]dt,$$

where γ_i is the intensity of the Poisson process describing the actual sales when the level of goodwill is i. Using the Pontryagin maximum principle, Rishel (1985) shows that the open-loop optimal advertising policy in this partially observed model consists of bang-bang and singular segments (see Sethi and Thompson 1981a) and can be obtained explicitly as

$$u^* = \min\left[\bar{u}, \max\left\{0, \frac{\sum_{i=0}^{\infty} (\gamma_{i+1} - \gamma_i)\pi[i\mu P_i - (i+1)\mu P_{i+1}]}{\sum_{i=0}^{\infty} (\gamma_{i+1} - \gamma_i)\pi(\lambda P_{i-1} - \lambda P_i)}\right\}\right].$$

Rishel (1985) also treats the special case with $\gamma_0 = 0$ and $\gamma_i = \gamma$, $i \ge 1$. In this case, it is easily shown that the optimal policy is initially to advertise at the maximum possible rate to build up goodwill to the desired level and then spend just enough to maintain it there. Such a policy results also in the deterministic counterpart of the Rishel model. Recall that this policy is of the same type as in the NA model discussed in §2.1.

Of course, the open-loop assumption, made largely for analytical tractability, is a serious drawback. It would be preferable to have a closed-loop solution, in which the advertising rate will depend on the past values of observed sales. Unfortunately, an explicit formula for the closed-loop situation cannot be obtained, and, therefore, one must resort to numerical means to obtain a solution.

2.5. Product Quality Models

The first control-theoretic marketing model which includes quality is that of Kotowitz and Mathewson (1979a, b). Since it has already been reviewed in detail by Sethi (1977a), we shall not discuss it here.

First, we discuss a model due to Spremann (1985), which is based on the important observation that markets for certain nonstandardizable services, insurance and credit contracts, work force, and second-hand consumer durables are often characterized by *asymmetrical information* with respect to both quality and price. He mentions medical services, health resorts, unfranchised restaurants, and new vacation sites as some of the examples. Before their first visit to the service establishment or purchase, consumers do not know exactly the quality of the service. The quality of such *experience goods* (or services) can be assessed only after consumption; see Nelson (1970). Interested consumers communicate with others who have already used the service, and each customer served or each unit of the product sold carries a market signal. Spremann (1985) assumes the seller's *reputation* as a second signal whose intensity depends on two factors: the *quality-price ratio* as perceived by former customers and the *quantity* already sold.

Spremann assumes that sales depend on the current reputation $R(t)$, the price $p(t)$, and the stock of advertising goodwill $A(t)$ as follows:

$$S(t) = p(t)^{-\eta} A(t)^{\omega} \phi(R(t)), \tag{2.9}$$

where $\eta > 1$ denotes the elasticity of demand with respect to price, $\omega \in (0, 1)$ is the elasticity with respect to the stock of advertising goodwill, and the positive, strictly increasing function ϕ converts reputation into sales.

Goodwill $A(t)$ is formed according to the NA dynamics (2.1). Reputation $R(t)$ increases in proportion to the output of the firm and to the degree of the consumer satisfaction depending on the quality-price ratio. This is the ratio $p^0/p(t)$, where $p(t)$ is the price actually charged and p^0 denotes the reference price or the price consumers would be willing to pay for the given level of quality. This leads to the second state equation

$$\dot{R} = h(p^0/p)S - \gamma R, \quad R(0) = R_0, \tag{2.10}$$

where the *experience function h* describes the effect of the quality-price ratio upon customers' judgements. $\gamma > 0$ represents the rate of depreciation of reputation over time.

The experience function h is monotonic with

$$h(p^0/p(t)) \begin{Bmatrix} < \\ \ge \end{Bmatrix} 0 \quad \text{for} \quad p^0 \begin{Bmatrix} < \\ \ge \end{Bmatrix} p(t),$$

$$\text{i.e., for services} \begin{cases} \text{that are too expensive,} \\ \text{of good value.} \end{cases}$$

Consequently, $R(t) > 0$ means that word-of-mouth enhances reputation and demand, in turn (see (2.9)). When $R(t) < 0$, what people are saying is working against the business. Finally, $R(t) = 0$ acts as the neutral level. Note that $R(t) \equiv 0$ results in the NA model.

Now the problem of investment in the two assets *goodwill* and *reputation* can be stated as the following optimal control problem:

$$\max_{p,u} \int_0^\infty e^{-rt}[(p-c)S-u]dt$$
$$\text{s.t. } (2.1), (2.9), (2.10), \tag{2.11}$$

where $r > 0$ is the discount rate and $c > 0$ denotes unit production cost.

The focus of Spremann's paper is the question of what price the firm should charge or, indirectly, how much should the firm invest in its stock of reputation. It turns out that there are two optimal policies for the firm: (i) charging lower than the reference price resulting in positive recommendations from the customers, and (ii) charging higher than the reference price so that customers have good reason to advise against further visits, thus reducing future demand. Both equilibria are considerable departures from the price-corresponds-to-quality mode which is not optimal if reputation is being generated. Moreover, Spremann shows that the firm's optimal advertising effort in relation to its sales is proportional to its optimal price level. Thus, in a situation, where positive market signals are spread as output increases ($h > 0$), the firm reallocates the promotional effort from advertising in favor of price cuts. If, on the other hand, the word-of-mouth is negative ($h < 0$), the accumulation of reputation must be reduced in favor of advertising.

The system dynamics of Spremann imply that prices higher than the reference price depress the sales rate, which, in turn, slow down the decline in reputation. To avoid this effect, Feichtinger et al. (1988) assume that purchase decisions are typically taken in two stages. They consider a retailer that deals with convenience goods, e.g., a supermarket. Convenience goods are purchased by consumers without much stress on comparison and choice. At the first stage the prospective customer decides on the particular store to go to, and then, once in that store, he decides which items and how much of each to buy. While the first decisions will rely on past experience, advertising, and word of mouth information, the latter ones can be based on actually observed prices. Single observation of prices will usually not be remembered for long; instead they tend to be amalgamated into a general idea of the store's price image or reputation.

To formulate the model of Feichtinger et al. (1988), let $\mathbf{p}^0$ denote the vector of reference prices that corresponds to the vector $\mathbf{p}(t)$ of actual prices, and $\mathbf{h}$ a vector of weights that consumers attribute to the various products when assessing the price image of a store. Then the scalar product $\mathbf{h}'[\mathbf{p}^0 - \mathbf{p}(t)]$ is an aggregate information on how the store's actual prices compare to the reference prices at time t. This gives the following differential equation for the reputation R:

$$\dot{R} = \mathbf{h}'(\mathbf{p}^0 - \mathbf{p})v - \gamma R, \tag{2.12}$$

where γ has the same meaning as in (2.10) and $v(t)$ denotes the number of shoppers visiting the store per unit time. Thus, the information to be learned enters the model weighted by the number of persons *exposed* to it and not by sales as in Spremann. This feature distinguishes the approach of Feichtinger et al. also from other adaptive learning models as those of Schmalensee (1978), Kotowitz and Mathewson (1979a, b), and Conrad (1985).

Feichtinger et al. assume that $v(t)$ is a function of the current advertising outlay $u(t)$ and the current reputation $R(t)$. Specifically,

$$v = v(u, R) = u^\beta \phi(R) \tag{2.13}$$

with a constant advertising elasticity $\beta \in (0, 1)$ and a strictly concave reputation response function ϕ. Furthermore, it is supposed that the price response of the visitors in the store is linear. Thus

$$\mathbf{s} = \mathbf{M}(\bar{\mathbf{p}} - \mathbf{p})v(u, R), \tag{2.14}$$

where $\mathbf{s}$ denotes the vector of sales rates and $\bar{\mathbf{p}}$ denotes the vector of backstop prices at which the demand vanishes. $\mathbf{M}$ is a quadratic matrix of the response coefficients m_{ij} measuring the influence of the price for item j on the sales of item i; $m_{ij} > 0$ means that a price cut for item j leads to an increase in the sales of i.

Considering reputation as the only goodwill capital, the authors obtain a one-state optimal control model:

$$\max_{p,u} \int_0^\infty e^{-rt}[(\mathbf{p}-\mathbf{c})'\mathbf{M}(\bar{\mathbf{p}}-\mathbf{p})v(u,R) - u]dt$$

subject to (2.12) and the initial condition $R(0) = R_0$, where $\mathbf{c}$ denotes the vector of unit production costs.

For this model, it is possible to obtain the behavior of the optimal price and advertising policies. It turns out that increase in reputation is achieved mainly through pricing; increased advertising follows only after the reputation has been improved. Therefore, as long as the reputation is poor relative to the optimal equilibrium level, a larger portion of the profit should be allocated to lower prices rather than to increased advertising. On portions where the reputation declines towards the equilibrium, however, increased advertising is profitable.

Conrad (1985) also considers an asymmetric information situation where the quality of a good can be evaluated by consumers only after purchase. The objective of the producer is to maximize profits by choosing optimal quantity, quality, and the advertising level for the product. The quality, expected by the consumer on the basis of goodwill, is known to the producer. If goodwill has promised a higher quality than the consumer experiences after purchase, the goodwill will be reduced. If, on the other hand, the actual quality exceeds the expected quality, this will increase the producer's reputation. Using a phase diagram analysis, Conrad characterizes the optimal paths of quality, quantity, advertising, and goodwill.

One of his conclusions is that producers either permanently signal a higher quality than what is produced, thus reducing the initial goodwill or they offer a higher quality than what the consumers expect and thus build up goodwill. Moreover, quantity and advertising will increase and quality will decrease monotonically over time if goodwill is low initially.

Table 2.1 presents a summary of the quality models described above along with the quality models discussed in §3.4.

2.6. General Remarks

Now that various models have been reviewed in which goodwill or reputation of a firm is formed by its marketing decisions, it would be to useful to make some general remarks in order to conclude the section. The first class of models, namely the advertising capital (goodwill) models of §2.1, represents earliest modelling attempts using optimal control theory in the area of advertising. While the models themselves are not considered very realistic, they do form the basis of many of the models in the literature. Models briefly discussed in §2.2 introduce a limited advertising budget into the goodwill models. It should be noted that the incorporation of a limited budget is not restricted to the goodwill

Table 2.1 Summary of the Product Quality Models

	State Variable	System Dynamics	Typical Qualitative Monotonicity Results
Kotowitz and Mathewson (1979a, b)	expected quality	$\dot{Q} = k(q - Q) + g(u)$	advertising u decrases over time, quality q decreases, sales s increase if Q is small initially
Spremann (1981, 1985)	goodwill A, reputation R	(2.1), (2.10)	advertising expressed as a fraction of sales is proportional to price
Feichtinger et al. (1988)	reputation R	(2.12)	advertising u increases, price p increases (if R is small initially and not too far from the long run equilibrium)
Conrad (1985)	goodwill A	$\dot{A} = q - Q(u, A)$	advertising u increases quality, q decreases, sales increase (if A is small initially and $Q_A > 0$, $Q_u < 0$)
Tapiero (1981)	sales s	$\dot{s} = u - \delta(q)s$	u decreases (if s is small initially); q increases (if $r + \delta(q)$ is large and the profit margin is low)
Ringbeck (1985, 1986a)	market potential x	$\dot{x} = g(u)(1 - x) - \delta(u, q)x$	u decreases, q increases (if x is small initially)

models and could be certainly considered in connection with other models reviewed in this paper.

The importance of lag models in §2.3 can be seen in two ways. On the one hand, they represent an interesting modelling approach based on different lag structures present in dynamics of goodwill and advertising.

On the other hand their autoregressive dynamics facilitates a better fit with empirical data and allows us to understand the actual relationship between goodwill (or sales) and advertising.

A common feature of many product quality models reviewed in §2.3 is that the dynamics depends critically on the actual quality and the quality expected by consumers. Different patterns of optimal advertising rates occur depending on the way quality is included into the model. It is important to note that there is a serious lack of research devoted to empirical validation of these models. Perhaps, these models could be extended to incorporate lagged effects and then estimated. Of course, this presumes a great deal with respect to the availability of suitable data.

3. Sales-Advertising Response Models

These models are characterized by a direct relation between the rate of change in sales and advertising in the form of a differential equation. Pre-1977 models have already been treated in Sethi (1977a). To discuss the subsequent models, it would be convenient to reproduce the classical Vidale-Wolfe (1957) (VW, hereafter) advertising model

$$\dot{x} = \rho u(1 - x) - \delta x, \quad x(0) = x_0 \geq 0, \qquad (3.1)$$

and the Ozga (1960) model

$$\dot{x} = \rho u x(1 - x) - \delta x, \quad x(0) = x_0 > 0, \qquad (3.2)$$

where x is the captured fraction of the market potential, ρ is the advertising response constant, and δ is the decay constant (cf. also Stigler (1961)). The dynamics (3.1) and (3.2) are fundamentally different from the NA dynamics (2.1) because of the presence of the "diffusion" terms $\rho u(1 - x)$ and $\rho u x(1 - x)$ in place of the term u, which reflects the investment in the stock of goodwill in (2.1). As can be seen below and in Sethi (1977a), this difference has significant consequences for the results that follow from sales-advertising response models.

The optimal advertising policies for a profit-maximizing firm whose market dynamics is given by (3.1) or (3.2) are reviewed by Sethi (1977a). Both linear and nonlinear advertising cost functions are considered.

3.1. Multiple Equilibria

While Gould (1970) had analyzed the problems both with VW and Ozga dynamics and convex advertising cost, his treatment was not exhaustive. More specifically, he obtained a single stable equilibrium or long run optimal sales level in each of the problems, which the system would converge to, from some initial sales level.

As it turns out, problems with the Ozga dynamics admit multiple stable equilibria and convergence to a particular equilibrium depending on the initial level of sales. In the linear objective function case, Sethi (1979a) has shown that there may arise up to three possible equilibria $x = 0$, $x = \hat{x} \in (0, 1)$, and $x = 1$. Feichtinger and Hartl (1986, §11.1.2) have obtained two equilibria, $x = 0$ and $x = \hat{x}$, in a nonlinear version of the model.

The consequences of having multiple equilibria are important to the firm. It means that the advertising policy depends critically on the initial sales level. Moreover, a firm with sufficiently small initial sales level would converge to $x = 0$ in the long run, if the parameters of the firm's problem yielded $x = 0$ as a possible equilibrium; not a rosy picture for the firm to *be* in the market under consideration.

3.2. Ad Wearout and Copy Replacements

Pekelman and Sethi (1978) address the problem of simultaneous decisions for the investment in the production of an ad, the advertising expenditures over time, and the timing for campaign or ad replacements. The "life cycle" of the effectiveness of an ad is influenced by all three decisions. Field studies by Pekelman and Tse (1980) have shown that the more one invests, the better, on the average, the initial performance of the ad. After the initial performance is realized, a wearout phenomenon may take place where, depending on the creative approach and the product category, the advertising effectiveness declines. Based on these empirical findings, Pekelman and Sethi develop their complex model, which is specified as follows.

Pekelman and Sethi denote the discrete times of copy replacements by

$$0 = T_0 < T_1 < T_2 < \cdots < T_m = T.$$

Let I_i denote the amount of investment in the copy design for the ith advertising campaign. They assume the VW dynamics (3.1) to hold but with a random advertising effectiveness coefficient ρ, which depends on the investment in the copy design and which decreases over time due to the ad wearout phenomenon. Thus, the sales x_i in the interval $[T_i, T_{i+1}]$ change, subject to

$$\dot{x}_i = \gamma(t - T_i, I_i)u_i(1 - x_i) - \delta x_i, \quad (3.3)$$

where u_i is the advertising rate and

$$\gamma(t - T_i, I_i) = \xi(I_i) - \mu(t - T_i, I_i);$$
$$d\xi/dI > 0, \quad \mu(0, I_i) = 0, \quad \partial\mu/\partial t \geq 0.$$

Here the initial performance $\xi(I_i)$ depends on the amount invested and is usually not known in advance. However, since a copy is usually pretested before it is launched, a finite number of possible values can be determined and the corresponding probabilities can be estimated. Furthermore the sales rate is continuous, i.e.,

$$x_i(T_i) = x_{i-1}(T_i);$$
$$i = 1, \ldots, m-1; \quad x_0(0) \text{ given}. \quad (3.4)$$

For simplicity it is assumed that the profit rate is $\pi x_i - u_i$. Thus the problem becomes

$$\max_{m,T_i,I_i} \mathbb{E}\left\{ \max_{u_1,\ldots,u_{m-1}} \sum_{i=0}^{m-1} \left[\int_{T_i}^{T_{i+1}} e^{-rt}(\pi x_i - u_i)dt - e^{-rT_i}I_i \right] + e^{-rT}\bar{S}(x_{m-1}(T)) \right\} \quad (3.5)$$

subject to (3.3), (3.4), and the control constraint $0 \leq u_i \leq \bar{u}$, for all i.

Pekelman and Sethi propose a mixed optimization technique using the maximum principle and dynamic programming for solving this stochastic problem numerically.

3.3. Wiener Process Models

In addition to the models described in §§3.2 and 2.4, not much has been done to incorporate stochastic considerations in optimal control models in advertising. Yet the problems in marketing contain significant uncertainties. Perhaps there is a need for a new, powerful methodology to be applied. An important candidate is the stochastic optimal control formulation with Wiener process. The formulation has been very popular in the areas of production and finance; see, e.g., Karatzas et al. (1986), Fleming et al. (1987), Tapiero (1988), Duffie (1988), and Karatzas and Shreve (1988). It is possible that models with Wiener processes might also yield successful results in marketing problems. To date, however, there have been only a few advertising applications incorporating Wiener processes; see also §6.3 for a model of a hierarchical marketing-production system that incorporates a semimartingale, which is more general than a Wiener process. Because of its modeling potential, we describe the stochastic optimal control models in advertising in general and review these applications.

Specifically, we would like to maximize

$$\max_u \mathbb{E} \int_0^\infty e^{-rt}\Pi(x, u)dt \quad (3.6)$$

subject to the stochastic differential equation

$$dx(t) = g(x, u)dt + \sigma(x, u)dW(t). \quad (3.7)$$

For technical assumptions required to rigorously state the problem and its solution methodology, the reader is referred to Fleming and Rishel (1975).

In (3.6) and (3.7), x denotes a stochastic process representing sales and $\Pi(x, u)$ is a concave profit function, the expectation of whose present value over an infinite horizon must be maximized. The mean evolution of x is given by (3.7) with $\sigma \equiv 0$. Function g is known as the drift of the x process, σ^2 expresses the instantaneous variance of the x process, and $W(t)$ is the standard Wiener process.

Raman (1988) introduces other control variables such as price p and quality q in (3.6) and (3.7). Then the first-order necessary conditions for an interior optimal solution for the control variables give rise to the Dorfman-Steiner (1954) type results.

Tapiero (1988) analyzes models with drift g given by either the NA model (see also Raman 1990) or the VW model. He provides a numerical solution for the problem in the VW case. Moreover, he also relates the optimal advertising policy to the firm's risk aversion index, given by $(\partial^2\Pi/\partial u^2)/(\partial\Pi/\partial u)$.

Sethi (1983a) assumes

$$\Pi(x, u) = \pi x - u^2, \quad g(x, u) = \rho u\sqrt{1-x} - \delta x,$$
$$\sigma(x, u) = \sigma(x)$$

satisfying $\sigma(x) > 0$, $x \in (0, 1)$, and $\sigma(0) = \sigma(1) = 0$. It should be noted that the assumed g is a variant of the VW dynamics. The variation is motivated by the consideration that an explicit solution can be obtained. It is interesting to note that the optimal solution consists of an equilibrium level $\hat{x}$ below which it is optimal to advertise at the maximum possible level, while it is optimal not to advertise above it. The form of this solution is similar to the bang-bang rule in the deterministic case. The important difference is that now the sales level is a diffusion process around the equilibrium level $\hat{x}$.

3.4. Product Quality Models

Ringbeck (1985, 1986a, 1987) studies the problem of a firm choosing optimal quality and advertising strategies under asymmetric information about product quality. His model is based on the VW model. Under special assumptions it is shown that it is optimal to enter the market with low product quality offering a low price and choosing a high advertising rate. As time goes on, the producer should increase quality and price but decrease advertising. It is interesting to note that due to different assumptions, this result is the reverse of that of Conrad (1985) described in §2.5 and of Kotowitz and Mathewson (1979a), who predict a decreasing quality policy starting with high quality, if goodwill is low at the beginning. The different results depending on the different assumptions are summarized in Table 2.1.

3.5. General Remarks

As already noted the sales-advertising response models reviewed in this section can be considered more realistic than the capital stock models of the previous section since it is certainly more easy to measure, estimate, or even define sales compared to "goodwill." In their basic formulation they are still rather simple but—as the NA model—they form the basis of many other approaches dealt with in this survey. In subsections 3.1 to 3.4 we have reviewed several extensions of the basic diffusion models. It does not seem appropriate to repeat these discussions here.

4. Cumulative Sales or Market Growth Models

With durable goods, the current demand is often influenced not only by price and advertising but also by the cumulative number of adopters through social influence processes; Bass' (1969) model is the earliest one emphasizing the importance of cumulative sales. With this latter influence process, two types of carry-over effects may be distinguished:

(1) *Positive carry-over effect*: The current sales of the product increase with past sales since every past buyer acts via word-of-mouth as a positive source of information about the product.

(2) *Negative carry-over effect*: The demand decreases with past sales because of negative word-of-mouth effects. Moreover, the remaining market potential for durable goods is reduced by each unit sold (saturation effect).

A second dynamic effect which can be taken into account in a cumulative sales model is the cost learning phenomenon, which states that the unit production costs decrease with cumulative output. The Boston Consulting Group (1972) observed that the total unit cost (in constant dollars) declines by a factor of 10% to 30% each time the accumulated production volume doubles. Thus, one can assume that the unit production cost is a nonincreasing function of cumulative production volume.

These models are known variously in the literature as cumulative sales models, market growth models, or diffusion models. We use the first two terms. The first term is obvious. The second term, used by Urban and Hauser (1980), puts the emphasis on the major marketing phenomenon taking place. We do not use the last term because of the confusion that arises from the fact that it has also been used in the literature (see Gould 1970) for VW and Ozga models specified in §3.

Let $X(t)$ denote the cumulative sales by time t. As before, $u(t)$ and $p(t)$ signify the advertising rate and the price at time t, respectively. The sales rate in general, as formulated in Teng and Thompson (1985),

$$\dot{X} = S(X, u, p), \tag{4.1}$$

depends on cumulative output, advertising rate, and price.

The effect of $X(t)$ on current sales may not be uniform but can change with the different stages of the product life cycle. Typically, it will be positive during the introductory phase of a new product, i.e., $S_X > 0$, and becomes negative in later stages when saturation effects dominate, i.e., $S_X < 0$. Moreover, assume that

$$S_u > 0, \quad S_{uu} < 0, \quad S_p < 0, \quad S_{pp} < 2S_p^2/S. \tag{4.2}$$

As mentioned above, the unit production cost c is assumed to be a nonincreasing function of the cumulative output, i.e., of the production experience of the firm:

$$c'(X) \le 0. \tag{4.3}$$

If r denotes the discount rate and T the length of a fixed planning horizon, the objective of the firm is to maximize its total discounted profit

$$\max_{u,p} \int_0^T e^{-rt}\{[p - c(X)]\dot{X} - u\}\,dt \tag{4.4}$$

subject to (4.1) and a given initial stock of cumulative sales $X(0) = X_0$.

We now review some typical results for the case of a separable *dynamic demand function*

$$S(X, u, p) = g(X, u)h(p), \tag{4.5}$$

which generalizes Kalish (1983) by incorporating advertising as an additional variable. Separability in this sense implies that the price elasticity of demand is independent of current advertising activities as well as of cumulative sales.

From the maximum principle and (4.2), Teng and Thompson obtain the monotonicity result

$$\dot{u}(t) \gtreqless 0 \quad \text{if and only if} \quad \partial\epsilon/\partial u \gtreqless 0, \tag{4.6}$$

for the undiscounted case ($r = 0$), where $\epsilon = Xg_X/g$ is the elasticity of the dynamic demand w.r.t. cumulative sales. This means that the optimal advertising rate is increasing if advertising becomes more efficient during the product life cycle. Moreover, it holds that

$$\dot{p} \gtreqless 0 \quad \text{if and only if} \quad g_X \gtreqless 0, \tag{4.7}$$

which means that if positive carry-over effects prevail in the introductory phase of the product life cycle, *penetration pricing*[1] is optimal, i.e., the price should start at a low level and increase.

A special case of (4.5) is the dynamic demand function

$$S(X, u, p) = [\bar{X}(u) - X]h(p), \tag{4.8}$$

where $\bar{X}(u)$ denotes the market potential assumed to depend on the current advertising u. For $r = 0$, optimal pricing follows a *skimming policy*, i.e., price starts at a high level and decreases, while optimal advertising increases during the whole life cycle.

Dockner et al. (1989) consider the following mixture of the models by Robinson and Lakhani (1975) and Horsky and Simon (1983):

$$g(X, u) = (\alpha_1 + \alpha_2 \ln u + \alpha_3 X)(\bar{X} - X),$$
$$h(p) = e^{-\alpha_4 p}, \tag{4.9}$$

where α_i and $\bar{X}$ are positive constants. For the dynamic demand function (4.5) with (4.9), they show that optimal advertising in the case of no discounting decreases over time. For the discounted case, they derive

$$\operatorname{sgn} \dot{p} = \operatorname{sgn} \dot{g}_u. \tag{4.10}$$

This result can be interpreted as follows. If advertising becomes more (less) efficient, i.e., $\dot{g}_u > 0$ ($\dot{g}_u < 0$), then price has to be increased (decreased). So, under these assumptions, skimming pricing corresponds to high introductory advertising, while penetration corresponds to low initial advertising.

Furthermore, for a separable dynamic demand of form (4.5) with positive discounting and no cost learning ($c'(X) = 0$), Dockner et al. show:

$g_X(X(T), u(T)) > 0$ implies $\dot{p}(t) > 0$ for $t \in [0, T]$. This means that penetration pricing is optimal, even if

[1] In this section we use the terms "penetration" pricing and "skimming." These terms can be defined as follows: *penetration pricing* means a pricing policy which tries to penetrate the market quickly by initially charging a price which is lower than the marginal cost (or than the statically optimal price). Loosely speaking this term is, however, sometimes used for any increasing pricing policy.

Skimming policy refers to a pricing policy which tries to skim the complete consumer surplus by selling the product to every customer exactly at the price he is willing to pay. This is done by initially charging a high price and decreasing it gradually over time.

positive carry-over effects prevail at the end of the life cycle;

$g_X(X(0), u(0)) > 0$ and $g_X(X(T), u(T)) < 0$ imply that the optimal pricing policy is initially increasing followed by price cuts towards the end of the life cycle;

$g_X(X(0), u(0)) < 0$ implies that $p(t)$ initially decreases and then increases. If carry-over is negative from the beginning, a skimming policy is optimal followed by a gradual increase in prices towards the end of the life cycle; and

$g_X(X(t), u(t)) < 0$ for all $t \in [0, T]$ and $Xg_X/g \geq Xg_{uX}/g_u$ imply $\dot{a}(t) < 0$ for $t \in [0, T]$.

Summarizing the above results, the multiplicative structure of the demand function implies that the price policy resembles, in a sense, the product life cycle. Furthermore, if advertising is efficient, it can be shown that price should be used to increase marginal revenue, but not to increase the demand rate. Only if advertising becomes less efficient, the firm should raise the demand by lowering its price.

In Table 4.1, various special cases of the dynamic demand function (4.1) are listed along with references. The table is divided into three parts consisting of models with both advertising and price, models with only advertising, and a selection of models with price only. An earlier survey about some of these models as well as about estimation of these dynamics can be found in Mahajan and Peterson (1985) and Kalish and Sen (1986).

Chatterjee and Eliashberg (1990) employ a micromodelling approach which considers the impact of information on the dynamics of the adoption process at the individual level. This is aggregated to obtain a market dynamics, which is used in an optimal control formulation to obtain normative guidelines for promotional policy.

Table 4.1 Special Cases of the Dynamic Demand Function (4.1)

Dynamic Demand Functions	References
$g(X, u)h(X, P)$	Teng and Thompson (1985), Dockner et al. (1989)
$g(X, u)[\bar{X}(p) - X]$	Erickson (1982)
$(\alpha_1 + \alpha_3 X)[\bar{X}(u, p) - X)$	Mahajan and Peterson (1985)
$(\bar{X} - X)(\alpha_1 + \alpha_2 \ln u + \alpha_3 X)e^{-\alpha_4 p}$	Dockner et al. (1989)
$g(X, u)$	Dockner and Jørgensen (1988)
$(\bar{X} - X)(\alpha_1 + \alpha_2 \ln u + \alpha_3 X)$	Horsky and Simon (1983)
$h(X, p)$, $g(X)h(p)$	Kalish (1983)
$(\alpha_1 + \alpha_3 X)(\bar{X} - X)h(p)$	Robinson and Lakhani (1975)
$(\alpha_1 + \alpha_3 X)[\bar{X} - X]$	Bass (1980)

To conclude this section, we cite Assmus et al. (1984), Simon and Sebastian (1987), Sultan et al. (1990), and Horsky (1990) as examples of papers that estimate the parameters of some of the durable goods models in Table 4.1 and empirically validate them.

4.1. General Remarks

The cumulative sales dynamics underlying the models in this section was first developed in the framework of pricing models and then subsequently generalized by including other marketing instruments such as advertising; see Table 4.1. We have chosen to present one lead model with price and advertising and to review some typical results. In the light of earlier surveys by Mahajan and Peterson (1985) and Kalish and Sen (1986) we did not devote much space to discussing various other variants of the basic model. One line for further research would be the extension of the models to include quality as an additional marketing instrument.

5. Models with More Than One State Variable in the Advertising Process

In the first subsection, different types of (potential) customers and the flows between these groups are considered. Then, in §5.2, the superiority of pulsing advertising policy compared to equally spread advertising efforts is discussed.

5.1. Trial/Awareness Advertising Decisions

The models presented in §3 are simple two-stage models, i.e., they consider the flow from potential adopters to current adopters. In practice, however, an adopting unit may pass through more stages. For the attempts made to extend two-stage models to incorporate the multistage nature of the diffusion process, see Lavidge and Steiner (1961), Midgley (1976), Dodson and Muller (1978), Kalish (1985), Jedidi et al. (1989), and the monograph by Mahajan and Peterson (1985).

Näslund (1979) considers a model representing advertising response by several interacting variables such as awareness, intention, trial, and repeat. As in Nicosia's (1966) model, the system dynamics is similar to a harmonic oscillator. Näslund obtains long run pulsing as an optimal policy for certain constellations of the model parameters. However, it was argued by Muller (1983) that a constellation as required will occur only fortuitously in marketing practice.

Muller (1983) presents a dynamic model of a new product introduction based on a diffusion process, which makes the distinction between two types of advertising objectives: increasing awareness *and* changing predisposition to buy. The market of size N is divided as

$$N = x_1(t) + x_2(t) + x_3(t) \quad \text{where} \tag{5.1}$$

$x_1(t)$—number of people who are unaware of the existence of the product,

$x_2(t)$—the number of potential customers who are aware of the product but have not yet purchased it, and

$x_3(t)$—the number of customers who have purchased the product.

In order to influence the transition probabilities between these three groups, the firm can choose two types of advertising:

$u_1(t)$—awareness advertising, which informs customers about the product and thus transfers them from the unaware group x_1 to the potential group x_2, and

$u_2(t)$—trial advertising, which persuades customers to purchase the product and thus transfers them from group x_2 into the current customers group x_3.

In view of (5.1), which defines $x_2 = N - x_1 - x_3$, the transitions of the customers between the different groups can be represented by the following equations:

$$\dot{x}_1 = -u_1 x_1 - k x_1 (N - x_1)/N, \quad x_1(0) = N, \tag{5.2}$$

$$\dot{x}_3 = (a + u_2)(N - x_1 - x_3) - \delta x_3, \quad x_3(0) = 0. \tag{5.3}$$

The interpretation of these equations is similar to the VW dynamics (3.1): The people who know, $N - x_1$, contact and inform a total of $k(N - x_1)$, out of which only a fraction of x_1/N are newly informed. Out of the total number of people informed via advertising, $u_1 N$, only a fraction of x_1/N are newly informed. The constant a is the first purchase rate which can be increased additionally by trial advertising. As usual, δ is the decay rate at which customers are purchasing rival brands or not purchasing at all. With b denoting the repeat purchase rate, the sales rate $S(t)$ is given by

$$\begin{aligned} S &= (a + u_2)(N - x_1 - x_3) + b x_3 \\ &= \dot{x}_3 + \delta x_3 + b x_3. \end{aligned} \tag{5.4}$$

The advertising expenditures needed to generate the awareness effect u_1 and the trial effect u_2 are denoted by $U_1(u_1)$ and $U_2(u_2)$, respectively. Muller (1983) assumes these to be convex functions. The objective is to maximize the present value of the profit stream

$$\int_0^\infty e^{-rt}\{pS - U_1(u_1) - U_2(u_2)\}\,dt, \tag{5.5}$$

subject to (5.2) and (5.3). Substituting for $S(t)$ from (5.4) into (5.5) and integrating by parts, the resulting term with $\dot{x}_3$, yields the new objective

$$\int_0^\infty e^{-rt}\{p(r + \delta + b)x_3 - U_1(u_1) - U_2(u_2)\}\,dt. \tag{5.6}$$

With the help of the maximum principle, Muller (1983) derives differential equations for the advertising rates u_1 and u_2. From these, the monotonicity properties of the two types of advertising rates can be obtained. One main result is that in the (u_2, u_1)-plane the trajectory cannot move counterclockwise. In other words, if there exists a solution which converges towards a steady state, then it must start at a point $u_1 > 0$, $u_2 = 0$ on the u_1-axis and end at a point $u_1 = 0$, $u_2 > 0$ on the u_2-axis.

While in many marketing models the optimal advertising rate decreases monotonically over time, this does not have to hold if different types of advertising are considered. Thus, awareness advertising can indeed decline over time, but since trial advertising is expected to increase, the total expenditure over time will depend on their sum. Thus, if u_1 is initially increasing, then the total expenditure TE $= U_1(u_1) + U_2(u_2)$ will increase. On the other hand, if u_1 is decreasing initially and thus decreasing throughout, then TE may increase or decrease over time.

Mahajan et al. (1984) extend the Muller model by splitting the groups of potential customers x_2 and of current customers x_3 into two subgroups

$$x_2 = x_2^+ + x_2^-, \quad x_3 = x_3^+ + x_3^-,$$

where the superscript "+" denotes the number of people in the group who spread positive information while the superscript "−" denotes those who spread negative information. Mahajan et al. (1984) applied their model to forecast attendance for the movie "Gandhi" in the Dallas area.

Another extension of the Muller (1983) model is analyzed by Jedidi et al. (1989). In their model, adoption is conditional on the level of awareness as well as a consumer's ability to purchase the new product. This ability is determined through the current product price and the income of the consumer. Only aware consumers whose income is greater than a "critical income" are considered as potential adopters of the innovation.

5.2. Pulsing Advertising Policies

There is some empirical evidence indicating that a pulsing advertising schedule is superior to equally spread advertising efforts in some circumstances (Simon 1982); see also Sethi (1971) for evidence to the contrary. During the last decade, there have been a few attempts to construct dynamic optimization models for which the optimal policy is a pulsing schedule. These are described below.

An important feature of the Näslund (1979) model is its linearity. In this type of model, long run pulsation dies out or explodes except in fortuitous cases in which advertising behavior is oscillating sinusoidally. To describe persistent periodic solutions, some nonlinearities have to be included.

According to Sasieni (1971), pulsing schedules occur when the response function is not concave. In such a case it turns out that chattering as fast as possible between high and low effort yields a better result than operating on the response function itself.

Recently, Mahajan and Muller (1986) (see also Mesak and Darrat (1992) and Mesak (1985)) have also attempted to explain the superiority of pulsation from an S-shaped advertising effectiveness function g: Marginal efficiency is increasing for low levels of effort, e.g., because of threshold phenomena, and decreases only beyond a fairly high effort rate. Let a budget B be allotted for a period of length T and let $\tilde{u} = \arg\max\{g(u)/u\}$. Then alternating between $\tilde{u}$ and zero advertising effort may be superior to an equal spending rate B/T, if B/T is on the nonconcave section of g. The phase lengths $\tilde{t}$ and t^0 of $\tilde{u}$ and zero advertising rates, respectively, must be chosen such that average outlay over the whole period remains the same, i.e.,

$$\tilde{u}\tilde{t} = (B/T)(\tilde{t} + t^0).$$

Mahajan and Muller suppose that advertising creates awareness y according as

$$\dot{y} = g(u)(\bar{y} - y) - \delta y,$$

where $\bar{y}$ denotes complete awareness of the entire population and δ is the forgetting rate. For such a model, clearly the phase lengths $\tilde{t}$ and t^0 should be as short as possible to get the maximum effect out of the advertising outlay. So, not unlike Sasieni (1971), *chattering* policy of switching back and forth instantaneously between 0 and $\tilde{u}$ would be optimal, and pulsing is conceived as the closest practical approximation to this policy. In this situation, it should be obvious that a high enough frequency of pulsing would beat the policy of equal spending. However, average performance of practically feasible pulsing, which lags behind instantaneous chattering, may well be inferior to equal spending. The object of Mahajan and Muller is to compute this lag for alternative feasible pulsation patterns and to compare it to equal spending.

The comment by Little (1986) on Mahajan and Muller raises some doubts on the chattering concept of pulsing, especially "*if the response functions (will) stay the same as smaller and smaller time intervals are examined*." One might add: Will consumers really perceive a series of TV spots as pulsing if these are scheduled every other day? What seems to be missing is a model that takes into account if an ad will be perceived as the first of a new pulse or as continuation of an old series. There are two approaches to modelling of this aspect. One is due to Simon (1982) and Luhmer et al. (1988). The other is due to Feinberg (1992).

Feinberg (1992) introduces a filter to account for the fact that too frequent pulses might not be recognized by the consumers / viewers as different from an equally spread advertising campaign. He elaborates this idea in the context of the Sasieni (1971) model with response function

$$\dot{x} = g(x, u), \tag{5.7}$$

by modifying the model as follows:

$$\dot{x} = g(x, z), \qquad (5.8)$$

$$\dot{z} = \zeta(u - z). \qquad (5.9)$$

Here z denotes the filtered variable which the consumer perceives instead of the advertising rate u. The property of this exponential filter is that it yields the same constant output for a constant as well as chattering advertising input, while it yields a pulsing output for any nonconstant periodic input. Although no mathematical proof is given, Feinberg (1992) demonstrates numerically the superiority of pulsing over constant advertising policies.

It should be noted that the essential purpose of the filter in (5.9) is also to introduce an inertia in the dynamics, as it was for the adaptation level as the intermediate variable introduced by Luhmer et al. (1988); cf. (5.9) with (5.12) below.

According to Simon's ADPULS model (1982), on the other hand, advertising has two different effects, opposite in direction and different in time lag. First, advertising pushes sales; second, it diminishes perception of future advertising. According to this view, perception is intrinsically a dynamic process, as it relates to an adaptation level (Helson 1964) depending on the past history of the stimulus level. In so far as the actual stimulus level is above the adaptation level, one gets an extra effect.

Simon's model is in discrete time and the adaptation level is the last period's stimulus level. The optimal policy alternates between a high and a low level of advertising from period to period. So the length of the time-period underlying the model determines the length of advertising flights and pauses. Advertising practitioners are interested in determining these lengths from the model output. Also Simon's model leads only to a pulsation of the chattering variety. Therefore, Luhmer et al. (1988) propose a modified formulation in continuous time.

Let $A(t)$ denote the advertising stimulus level at time t. Clearly A cannot be changed over time without any inertia. Thus, Luhmer et al. consider again the NA dynamics (2.1) for $A(t)$. For the advertising effort u, constant upper and lower bounds are assumed, i.e.,

$$0 \le u \le \bar{u}. \qquad (5.10)$$

In a continuous time version, the adaptation level, $\bar{A}$, cannot be defined as the last period's stimulus level. Helson (1964) takes adaptation to be the geometric mean of a finite number of past stimulus levels for a series of experiments. Luhmer et al. (1988) choose exponential smoothing to generate the required mean and get

$$\bar{A}(t) = \zeta \int_{-\infty}^{t} A(\tau) \exp\{-\zeta(t - \tau)\} d\tau, \qquad (5.11)$$

or, equivalently,

$$\dot{\bar{A}} = \zeta(A - \bar{A}). \qquad (5.12)$$

The parameter $\zeta > 0$ represents the relative weight of more recent levels of advertising capital (or the speed of adjustment). Note that, according to (5.11) or (5.12), $\bar{A}(t)$ always lags behind $A(t)$.

Following Simon (1982), Luhmer et al. suppose sales S to respond to advertising according to

$$\dot{S} = g(A) - \alpha S + w(A - \bar{A}), \qquad (5.13)$$

where α denotes the proportion of customers switching to other brands per unit time, $g(A)$ denotes the level effect of advertising, while $w(A - \bar{A})$ is the extra effect gained if the stimulus level is above the adaptation level. For simplicity it is assumed that

$$w(A - \bar{A}) = \bar{w} \max\{0, A - \bar{A}\} \quad \text{and}$$

$$g(A) = \nu \ln(A + 1),$$

where ν is a positive constant.

We are thus led to the following optimal control problem:

$$\max \int_0^\infty e^{-rt}(\pi S - u) dt, \qquad (5.14)$$

subject to the three state equations (2.1), (5.12), (5.13), the control constraint (5.10), and the initial conditions $A(0) = A_0$, $\bar{A}(0) = \bar{A}_0$, $S(0) = S_0$. Furthermore, r denotes the discount rate and π the gross profit per unit sold. Note that according to (2.1), inertia for the motion of A is necessary to avoid chattering. So A is a state variable in Luhmer et al. rather than the control variable as in Simon.

The problem is linear and can be shown to have an optimal solution. Since w has a kink at $A - \bar{A}$, the problem is nondifferentiable in the state and a *generalized* maximum principle has to be used. For special sets of parameters, a *cyclical* solution can be obtained numerically. The cycle consists of four segments. Beginning with a starting point on the cycle where $\bar{A}$ is the lowest, one traverses repeatedly the segments in the following order: segment of increasing A and $\bar{A}$, segment of decreasing A and increasing $\bar{A}$, segment of decreasing A and $\bar{A}$, segment of increasing A and decreasing $\bar{A}$. At the end of each cycle, one returns to the starting point and the process repeats.

The main result is that long run pulsation can occur as an optimal policy for the continuous time reformulation of Simon's ADPULS model. As opposed to the original discrete time version, the durations of advertising and nonadvertising phases result endogenously from the model rather than being needed as inputs.

A different approach is chosen by Hahn and Hyun (1991). They consider the one-state advertising model by Mahajan and Muller (1986) and analyze the effect of different cost on the optimal advertising policy. In particular, they show that the introduction of pulsing costs makes pulsing optimal under a reasonable condition.

Park and Hahn (1991) formulate a passive rival model (see §7.2 for other such models) in which the firm's market share follows a discrete-time version of the Lanchester dynamics that incorporates competitive effects of its rival's advertising expenditures. They show that pulsing can be superior even when the change in market share is a concave function of advertising. Moreover, if it is linear or convex, then it is beneficial for the firm to advertise at the high (low) level when its rival advertises at the low (high) level provided that the firm has stronger brand power, greater net pulsing effect, or a more convex advertising response than its rival. Mathematically, these results are hardly surprising. What is interesting is that Park and Hahn claim their findings to be in accordance with the suggestions of many managerial studies cited in their paper.

Table 5.1 summarizes the literature on pulsing advertising strategies.

5.3. General Remarks

Beginning with the model of Sasieni (1971) there has been continuous endeavors to construct models explaining the pulsing nature of advertising outlays, which occurs often in practice. Mathematically, S-shaped reaction functions measuring the influence of advertising expenditures on sales imply what are known as chattering policies. Chattering can not be equated, however, to pulsing. As a matter of fact, optimality of pulsing policies arises from multiple state variables. An important model in this regard is Simon (1982), based on the accumulation of an adaptation level. While in Simon's discrete approach the cycle length is exogenously determined by the discretization factor, it is endogenously derived in the continuous time model by Luhmer et al. (1988). However, an empirical validation of the continuous ADPULS model would be desirable.

6. Interaction with Other Functional Areas

In the models surveyed so far, only the marketing sector of the firm was considered and other sectors (such as finance or production) were either disregarded or introduced in the problem only in an indirect manner. For example, the production sector was represented merely by the production cost function and the finance sector merely by the presence of the discount rate.

On the other hand, there is an extensive literature on production problems in which the marketing sector is disregarded or unduly simplified. However, marketing and production policies in a firm are interdependent. Marketing policies are normally designed to generate demand for the firm's products, whereas production policies are normally designed to meet the demand. Therefore, production and marketing decisions need to be coordinated in order to achieve an optimal strategy.

Approaches which take other functional areas along with advertising into account are presented below. Marketing models that deal with other functional areas without considering advertising such as Thompson et al. (1984) and Eliashberg and Steinberg (1987) are excluded.

6.1. Decentralized Marketing-Production Planning

Abad (1982a, b) proposes a model for which the two functions *marketing* and *production* are represented by the Holt et al. (1960) model of production planning and the VW model of the sales-advertising relationship, respectively. Let

Table 5.1 Chattering and Pulsing Advertising Policies

	State Variable	System Dynamics	Optimal Advertising Policies
Sasieni (1971)	sales x	$\dot{x} = g(x, u)$ g nonconcave	chattering
Mahajan and Muller (1986)	awareness y	$\dot{y} = g(u)(\bar{y} - y) - \delta y$ g S-shaped (nonconcave)	chattering
Tapiero (1978) Sethi (1979*b*)	mean goodwill A, variance in goodwill V	$\dot{A} = u - \delta A$ $\dot{V} = u + \delta A - 2\delta V$ Random walk based dynamics	Sinusoidal convergence to equilibrium possible
Näslund (1979)	units sold x_1, consumer attitude x_2, motivation x_3	$\dot{x}_1 = b(x_3 - \beta x_1)$ $\dot{x}_2 = a(x_1 - \alpha x_2) + cu$ $\dot{x}_3 = mx_2$ linear	Long run behavior may be oscillating sinusoidally in hairline cases
Simon (1982)	goodwill A_t	discrete time version of (5.13)	pulsing optimal; reduces to chattering in continuous-time
Luhmer et al. (1988)	goodwill A, adaption level $\bar{A}$, (Smoothed or filtered goodwill), sales S	$\dot{A} = u - \delta A$ $\dot{\bar{A}} = \zeta(A - \bar{A})$ $\dot{S} = g(A) - \alpha S + w(A - \bar{A})$	pulsing optimal
Feinberg (1992)	sales x Filtered advertising z	$\dot{x} = g(x, u)$ $\dot{z} = \zeta(u - z)$	pulsing better than constant spending
Hahn and Hyun (1991)	awareness y	$\dot{y} = u(\bar{y} - y) - \delta y$, pulsing costs	pulsing optimal
Park and Hahn (1991)	market share x_t	$x_t - x_{t-1} = g(u_t)[1 - x_{t-1}] - \delta(u_t')x_{t-1}$	pulsing better than constant spending

$S(t)$ = sales rate in dollars per unit time at time t,
$I(t)$ = level of inventory at time t,
$v(t)$ = rate of production at time t,
$u(t)$ = the rate of advertising at time t

C_v = per unit cost of raw materials, direct labor, and other production costs that are proportional to $v(t)$,

$C_p[v(t) - v^*(t)]^2$ = rate of costs that are related to the deviation of the actual rate of production, $v(t)$, from the desired rate of production, $v^*(t)$, e.g., undertime/overtime costs, and

$C_I[I(t) - I^*(t)]^2$ = rate of costs associated with the deviation of the actual level of inventory, $I(t)$, from the desired level of inventory, $I^*(t)$.

Consider the following control problem

$$\max_{u,v} J = \int_0^T [(1 - \beta)S - u - C_v v - C_p(v - v^*)^2 - C_I(I - I^*)^2]dt + b_1 S(T) + b_2 I(T) \quad (6.1)$$

subject to

$$\dot{I} = v - S/p, \quad I(0) = I_0, \quad (6.2)$$

$$\dot{S} = \rho u(1 - S/M) - \delta S, \quad S(0) = S_0, \quad (6.3)$$

$$0 \le u(t) \le \bar{u}, \quad v(t) \ge 0, \quad (6.4)$$

where δ is the sales decay constant, ρ is the sales response constant, M is the saturation level of sales rate, p is the selling price assumed to be constant, β is the fraction reflecting all other variable costs, b_1 is the value of a unit sales rate at $t = T$, b_2 is the value of a unit inventory at $t = T$, and $\bar{u}$ is the maximum rate of advertising that the firm can effectively maintain. Equation (6.3) is the VW model; note that $0 \le S(t) \le M$ is satisfied automatically on account of (6.3).

In Abad (1982a), this model is solved in the classical way by applying Pontryagin's maximum principle to (6.1)–(6.4); see also Yang et al. (1987) who propose a modification of the above model by respecifying (6.3).

Also by varying $v^*(t)$, capacity expansion problems can be treated.

The same model is solved in Abad (1982b) using a decentralized procedure. The problem is split into two subsystems, the marketing problem (MARK)

$$\max_u \int_0^T [(1-\beta)S - u]dt + b_1 S(T) \quad (6.5)$$

$$\text{s.t.} \quad \dot{S} = \rho u(1 - S/M) - \delta S,$$
$$S(0) = S_0,\ 0 \le u \le \bar{u}, \quad (6.6)$$

and the production problem (PROD)

$$\max_v \int_0^T [-C_v v - C_p(v - v^*)^2 - C_I(I - I^*)^2]dt + b_2 I(T) \quad (6.7)$$

$$\text{s.t.} \quad \dot{I} = v - Z, \quad I(0) = I_0,\ v \ge 0. \quad (6.8)$$

It is clear that the two subsystems are linked only by

$$Z = S/p, \quad (6.9)$$

and that the sum of (6.5) and (6.7) yields the global objective (6.1).

However, in the above formulations, the optimal solutions of (MARK) and (PROD) will not yield the overall optimal solution. In particular the marketing department would sell too much since (6.5) ignores the cost of production. Thus, a coordination mechanism will have to assign a transfer price μ to the quantities passed between the departments MARK and PROD of the firm.

This is done by applying the usual maximum principle conditions to problem (6.1)–(6.4) with (6.2) replaced by the equation parts of (6.8) and with the additional equality constraint (6.9). From these conditions, one can modify the objective functions (6.5) of MARK and (6.7) of PROD as follows:

$$\max_u \int_0^T [(1-\beta)S - u - \mu S/p]dt + b_1 S(T), \quad (6.5')$$

$$\max_v \int_0^T [-C_v v - C_p(v - v^*)^2 - C_I(I - I^*)^2 + \mu Z]dt + b_2 I(T), \quad (6.7')$$

where μ is the adjoint variable associated with (6.9). Since these modified problems have μ and Z as (exogenous) input variables, we shall denote them by MARK(μ) and PROD(μ, Z). It turns out that the optimality conditions of the global problem are automatically satisfied by optimal solutions of MARK(μ) and PROD(μ, Z). In order to guarantee the overall optimal behavior of the firm, the constraint (6.9) and the optimality condition for the transfer variable Z have to be satisfied. As a result, it is possible to show that

$$\mu = C_v + 2C_p(v - v^*) \quad (6.10)$$

provided that $v > 0$. The global problem can then be solved by using a decentralized solution procedure that solves MARK(μ) and PROD(μ, Z) successively until (6.10) is satisfied.

Abad (1982b) solves the problem using a decentralized procedure (see Singh (1980) in this connection) and reports that after eight iterations of the procedure the optimal solution was obtained with sufficient accuracy. The optimal solution consisted of an initial interval in which $u = 0$ or $u = \bar{u}$ depending on whether S_0 is large or small. In a second interval the singular solution for the advertising rate was chosen. At the end, once again $u = 0$ or $u = \bar{u}$ depending on whether the terminal value b_1 of sales was small or large, respectively. Since the production cost was assumed to be strictly convex in v, the optimal production rate was clearly continuous.

Besides the resulting optimal solution, an important outcome of the solution procedure from the managerial viewpoint is the transfer price μ, which defines the value of one unit of the produced good that is transferred from PROD to MARK. This price is useful for internal accounting purposes.

6.2. Other Decentralized Models of the Firm

Abad (1987) extends model (6.1)–(6.4) by introducing the stock of labor as a second state variable in the production sector. Furthermore a third sector, the finance subsystem, is considered with two more state variables, cash and debt. Finally, the VW dynamics (6.3) is replaced by the NA dynamics. A decentralized procedure as described in the previous subsection is used to obtain the optimal solution. Clearly, as the model is fairly complex, this can only be done numerically. Using a cyclical (seasonal) demand pattern, nontrivial time paths for cash, debt, goodwill, inventory, and work force are obtained.

A similar model with nonlinear NA dynamics is solved by Sorger (1986). He replaces the finance subsystem by a simpler investment subsystem. The overall model consists of four subsystems, namely, personnel planning, investment, marketing, and production and is solved by using, once again, a decentralized solution procedure.

All of the above models deal with a one-product firm. In Abad (1989), a multiproduct firm is considered, in which for each product, advertising, pricing, and production are considered as decision variables. For each product the stock of inventory is the state variable, while the marketing instruments act as flow variables determining the demand rate. There is no state variable such as goodwill or sales. This model is also solved numerically using the decentralized approach.

6.3. Hierarchical Controls in a Stochastic Marketing-Production System

Sethi and Zhang (1992) present an asymptotic analysis of a hierarchical marketing-production system with stochastic demand and stochastic production capacity. Demand S is modelled by a stochastic differential equation (see (3.7)), in which drift g is given by the NA model and in the noise term $\sigma dW(t)$, σ is assumed to be small and $w(t)$ can be generalized to a semimartingale, if desired. Production system is modelled by $\dot{I} = -\delta I + v - S$. The uncertainty in the production system arises from failure-prone machines, whose rates of breakdown and repair are assumed to be large, i.e., of the order of $1/\epsilon$, where ϵ is small. The objective of the system is to obtain production and advertising rates in order to maximize an expected profit over a finite horizon; see Sethi and Zhang (1993).

Because σ and ϵ are small, Sethi and Zhang are able to derive a simple deterministic problem associated with zero values for σ and ϵ. In this limiting problem, the random demand is replaced by its mean and the stochastic capacity process is replaced by average machine availability.

From the solution of this simpler problem, Sethi and Zhang are able to construct asymptotically optimal decisions for the original problem. Moreover, for these constructed controls, they are able to obtain an estimate of their deviation from optimality.

6.4. General Remarks

As a summary of this chapter one can say that the addition of other functional areas clearly makes the resulting model more realistic as compared to isolated marketing models ignoring production/inventory and finance aspects, respectively. Also, there is a large number of marketing models and production models in the literature while there are only very few models available which consider several functional areas. This opens a wide field for potential future work.

On the other hand, because of the complicated nature of the resulting optimal control problem (several state variables) it is usually not possible anymore to derive general qualitative results such as monotonicity. Since the problems have to be solved numerically it would be valuable to estimate the model using real data. This has not been done so far, and it is hoped that this void will be filled in future. The decentralized approach which yields the transfer price μ for purposes of pricing between the functional areas should be very useful in practical situations.

7. Competitive Models

Most of the models reviewed in the previous sections deal with a monopolistic firm. It is the purpose of this section to review dynamic, competitive models in advertising. As has already been mentioned, there existed a very small number of differential game models in advertising by the time of Sethi's (1977a) survey. A good number of differential game models in advertising have appeared since then to fill at least a part of the void; see e.g., surveys by Jørgensen (1982a), Feichtinger and Jørgensen (1983), Eliashberg and Chatterjee (1985), Rao (1990), and a book by Erickson (1991). Also, the special issues of *Journal of Marketing Research* (1985) on Competition in Marketing and of *Marketing Science* (1988) on Competitive Marketing Strategy indicate an increasing interest in incorporating competitive effects in modeling the response of firms' sales to such marketing variables as advertising and price. Still most of the research with a few exceptions has been restricted to open-loop, deterministic games, which are not always appropriate.

In our review of the competitive models, first we shall discuss in §7.1 a hazard rate model of Bourguignon and Sethi (1981), that models the competitive aspects by the presence of a threat of entry into the market by another firm. In §7.2, we discuss models in which the goodwill/sales-advertising dynamics of the firm contains exogenously given advertising expenditures on the part of its rivals assumed to be passive. In §7.3, an adaptive optimal control model in advertising is discussed. Here, inclusion of total industry advertising in the model recognizes the competitive elements in the market. In §7.4, we describe models in which the competitive situation is represented by reaction functions. Finally, in §7.5, we discuss very briefly the differential game models in advertising in view of the existing surveys and the book noted above.

7.1. Hazard Rate Models

These models are employed in situations, in which a firm must advertise in order to deal with a threat of entry by another firm. To formulate such a model, one defines, as in Kamien and Schwartz (1971), a function $F(t)$ denoting the probability that entry has occurred in the time interval $[0, t)$. The entry rate or the hazard rate at time t, which is defined to be the conditional probability of entry at time t, provided that no entry has occurred by that time, is given by the expression $\dot{F}(t)/[1 - F(t)]$.

Bourguignon and Sethi (1981) assume the hazard rate to be a function of price p, advertising u, and F. A special class of hazard rates is given by $h(p, u) \times (1 - F)^{n-1}$ yielding

$$\dot{F} = h(p, u)(1 - F)^n, \quad F(0) = F_0. \tag{7.1}$$

Here, $h_p \geq 0$, $h_u < 0$, and n is a parameter reflecting the nature of potential entrants, where specifically, $n = 1$ implies neutral potential entrants, since the hazard rate is independent of F. The cases with $n \in (0, 1)$ imply aggressive potential entrants, who hasten their entry when $F(t)$ is high. They prefer to take advantage of the situation of a high $F(t)$ by entering the industry before others do. Cautious potential entrants, on the other hand, are represented by $n > 1$. A potential entrant of this type delays his entry or decides not to enter at all when $F(t)$ is high. Thus, he avoids the risk of entering the market, that is ripe for another firm to undercut him by entering before him.

With (7.1) as the state equation, and

$$\max_{p,u} \int_0^\infty e^{-rt}[(1 - F)(\Pi_1(p) - u) + F(\Pi_2)]dt \tag{7.2}$$

as the objective, a deterministic optimal control problem is obtained. Here, $\Pi_1(p)$ is the revenue function assumed to be concave and Π_2 is the constant profit rate subsequent to an entry by another firm. Bourguignon and Sethi (1981) analyze this problem with the use of Pontryagin's maximum principle and obtain the following results under mild technical assumptions.

For $n < 1$, there exists a critical $\bar{F}$ such that if $F \geq \bar{F}$, the optimal policy is to forbid the entry of any competitor by setting p and u appropriately (a stay-out rule), while for $F < \bar{F}$, the optimal policy is to let F increase up to $\bar{F}$ by allowing for some limited probability of entry (a market-skimming rule). For $n > 1$, there also exists an $\bar{F}$ such that $F \leq \bar{F}$ gives the stay-out rule while for $F > \bar{F}$, the optimal policy is to let F increase to 1 by allowing for some probability of entry. Finally, for $n = 1$, the optimal policy is stationary and under it F gradually approaches 1.

7.2. Passive Rival Models

The crucial assumption of these models is that the goodwill/sales-advertising dynamics depends on its competitor's advertising which is exogenously given. One such model by Ireland and Jones (1973) was already discussed in Sethi (1977a).

Horsky (1977a) analyzes a model in which the market share response to advertising is formulated as a first-order Markov process. Assuming that the probabilities depend on the advertising goodwill accumulated by the firm and its competitors, the author obtains a nonlinear model which is validated by empirical data. Given the empirical findings, the optimal advertising policy is derived and some managerial implications are derived.

7.3. Adaptive Control Models

In all of the models reviewed thus far, it is assumed that the parameters of the models are known. If this is not the case, then parameters must be estimated over time as more information becomes available. What we then have are adaptive optimal control problems.

Pekelman and Tse (1980) formulate an adaptive optimal control advertising model. Their sales-advertising relationship is based upon empirical data and expresses current sales in terms of lagged sales and the ratio of the firm's advertising to the total industry advertising. Note that it is this ratio through which competition enters the model. Given some additional relationships for industry sales and competitive advertising along with a temporal evolution of a parameter in the state equation, the adaptive control rule is derived using a slight modification of the method developed by Tse et al. (1973); see also Pekelman and Rausser (1978). Moreover, Pekelman and Tse (1980) perform simulations to demonstrate the superiority of the adaptive scheme obtained by them.

7.4. Reaction Function Models

Reaction function models are an attempt to formulate a competitive situation by introducing a function that captures the behavior of the competitors in response to the action by the firm under consideration. However, the problem is not solved as a Stackelberg differential game, in which the competitors must also be modelled as active players optimizing their own respective objective functions. Earlier, Jacquemin (1973) introduced such a reaction function in a nonlinear extension of the NA model. This has been discussed in Sethi (1977a).

Since then, Bensoussan et al. (1978) have developed another model incorporating a reaction function in the lagged system dynamics (2.3). They consider two types of marketing dynamics. First, the dynamics describing how the effects of a firm's marketing decisions are spread over time, and secondly, the competitive reaction dynamics. They assume continuously lagged consumers' responses to advertising campaigns. The case of a leading firm anticipating its competitors' reactions over time is examined. By means of reaction functions for the competitors, the original differential game model is reduced to a control problem which contains integro-differential equations of motions reflecting the general forms of both, the carry-over effects of advertising and of the delayed reactions of followers. Using necessary conditions for optimality, the authors derive steady state solutions and give economic interpretations. The results represent dynamic generalizations of the Dorfman-Steiner theorem for a monopoly as well as for oligopolistic markets previously derived by Lambin et al. (1975). The multiplicative demand structure implies a monotonic relation between the leader's optimal actions and the speed and intensity of the followers' reactions. Some numerical results illustrate a special case of the model in the discrete time setting.

7.5. Differential Game Models in Advertising

In §6 of Sethi (1977a), several oligopolistic advertising models were reviewed. In these models, competition enters only implicitly by way of a ratio of the firm's advertising to total industry advertising as in §7.2 or via a competitor's reaction function as in §7.3. Sethi noted that the resulting mathematical models are not much different from the monopoly model which they generalize. He went on to say that . . . "what is missing are competitors who themselves are optimizers and not just reactors whose reaction function is known by the firm under consideration." The answer lies, of course, in differential game advertising models. In this connection, Tapiero (1979), who develops a nongame multifirms NA-type advertising model as well as a differential game model, cites the empirical study (using Quaker-Oats data) by Pekelman and Tse (1980), which refutes the conclusions of the simple nongame multifirms model.

Since 1977 and starting with Deal et al. (1979) and Deal (1979), a substantial number of papers dealing mostly with deterministic open-loop differential games in advertising have appeared. Because many of these have been already surveyed in Jørgensen (1982a), Feichtinger and Jørgensen (1983), Eliashberg and Chatterjee (1985), Rao (1990), and Erickson (1991), we have chosen not to review them here. What we have chosen to do instead is to review a few recent papers that deal with closed-loop solutions along with some empirical validation. These are Erickson (1992), Chintagunta and Vilcassim (1991, 1992), and Chintagunta and Jain (1990, 1992a, 1992b).

Chintagunta and Vilcassim (1992) use the (deterministic) Lanchester model of combat, which can be viewed as a competitive extension of the VW-dynamics, to represent the market share dynamics that capture the competitive shifts due to advertising expenditures by two market rivals and obtain both open-loop and closed-loop Nash equilibrium solutions; see also Chintagunta and Vilcassim (1991). Furthermore, they use ordinary least squares regression to estimate the model parameters for Coke and Pepsi using published secondary data for the period 1968–1981. They are then able to compare the equilibrium advertising levels to actual advertising levels as well as to optimal advertising levels obtained as in §7.2. Their results indicate that closed-loop strategies provide a better fit to the data and suggest therefore that using these strategies is a more appropriate way of capturing the dynamics of advertising competition.

Erickson (1992) also compares closed-loop equilibrium advertising strategies with the actual spending levels in the market for beer; see also Sappington and Wernerfelt (1985). In his approach, the actual observed advertising spending levels are assumed to be generated under the assumption that the firms are acting as Nash competitors. Thus, he estimates a system of simultaneous equations, including Nash equilibrium first-order conditions, in which advertising expenditures and market shares are treated as both dependent and independent variables; see also Gasmi and Vuong (1988), Hanssens et al. (1990), and Chintagunta and Jain (1992b) for the use of simultaneous equations approach in differential game contexts.

On the other hand, Chintagunta and Jain (1992a) formulate differential game models played by a manufacturer and a retailer in a single marketing channel; see also Jørgensen (1984) and Chintagunta and Jain (1990). Their formulation involves NA-dynamics for the goodwills of the members with the channel sales depending on these goodwills in a nonlinear fashion. They obtain both coordinated and uncoordinated closed-loop strategies and show that channel members exert more marketing effort in the case of coordination. Moreover, they derive a number of testable hypotheses providing a basis for predicting when coordination would take place in a dynamic context.

Raman (1991) formulates an oligopolistic model with n firms. Competition in his model enters only indirectly via the untapped market. Furthermore, he obtains closed-form expressions for closed-loop equilibrium strategies.

In an interesting article, Roberts and Samuelson (1988) develop a dynamic model of advertising competition in the U.S. cigarette industry. The empirical results reveal that advertising primarily effects the size of market demand and does not alter firm market shares. The market share aspects of oligopolistic rivalry are better captured through changes in the number of brands sold by the firms, rather than by changes in advertising. It turns out that firm actions in the cigarette market recognize and exploit the dependence of optimal future rival actions on current advertising, i.e., dynamic conjectural variations occur. The empirical findings show that oligopolistic advertising competition can be viewed as a prisoner's dilemma, with the resulting noncooperative equilibrium advertising levels *not* exceeding the joint-profit maximizing level. This is contrary to the common belief, but statistical evidence supports the conclusion that excessive advertising does not occur.

8. Conclusions and Implications for Future Research

In the conclusion of his survey, one of the points noted by Sethi (1977a) was that, ". . . *what we need are differential game models . . . to effectively and meaningfully deal with oligopolistic situations*." Section 7, the surveys by Jørgensen (1982a), Feichtinger and Jørgensen (1983), Eliashberg and Chatterjee (1985), Rao (1990), and the book by Erickson (1991), show that a large number of papers have been published in this area since then. However, with a few recent exceptions most of the research has been restricted to deterministic open-loop solutions, which are not always appropriate. Further work is needed to include more general information structures such as closed-loop and feedback.

At the present state of knowledge most of the optimal advertising models are rather small in comparison to real life. With few exceptions, the models assume single-product, single advertising medium, and single market. This was already noted by Sethi (1977a), and this critical remark is still valid for the literature published subsequently. To formulate more realistic problems, extensive theoretical as well as empirical work has to be done to come up with multivariate sales advertising models. Since a closed-form solution or even a qualitative characterization of the solution is almost impossible for larger models, numerical methods have gained growing importance. These will be essential to make real-life applications of optimal control models in advertising.

A third point mentioned in Sethi's survey was the considerable lack of stochastic dynamic optimization models in marketing. As shown in §§3.3 and 6.3, some interesting work has been done in the last decade. However, this field continues to be a promising area for future research.

In addition, we would like to emphasize several new initiatives taken during the years since 1977. First, the *product quality*, although it had been recognized prior to 1977, has since become an important marketing instrument in addition to the traditional advertising and price decisions (§2.5). Another major development concerns the *market growth models for durable goods* described in §4. Special features of these models are carryover effects related to the various stages of product life cycles and learning cost phenomena. On the technical side, efforts were made to construct dynamic advertising models that exhibit *pulsing schedules* as optimal advertising policies; see §5.2. Finally, some important steps have been taken (see §6) in the direction of integrating marketing into more comprehensive *corporate models*. These recognize the interaction of marketing with the other functions in the firm. Also decentralized hierarchical control approaches are used in obtaining optimal solution for these models.

The major suggestions for further research that flow especially from these new initiatives is that there is a need to build more realistic corporate models that incorporate production, marketing, R & D, and finance. Moreover, the product quality should have an important place in these models given that it is already an important consideration from the viewpoints of the first three functions of the firm listed above. In this connection, it may be noted that there currently exist optimal control models in the production literature that incorporate quality; see Fine (1986), for example.

Finally, to conclude this survey, we must emphasize once again that extensive empirical and econometric work needs to be carried out in order to support the assumptions underlying the dynamics of sales, advertising, price, quality, etc., used in the existing optimal control models in the marketing literature and the corporate models yet to be formulated. To this end, we provide a number of testable propositions, in the style of Eliashberg and Chatterjee (1985), that follow from some of the models surveyed here. Our list is deliberately intended to be only a sample, since other such propositions can be easily formulated from the papers reviewed in the survey.

PROPOSITION 1. *In a broad class of advertising models (e.g., for those in §§2.1, 2.3, 3), higher discounting rates imply smaller advertising expenditures and a smaller stock of goodwill in the long run.*

PROPOSITION 2. *If the time lag between advertising and the corresponding increase in goodwill obeys a general distribution, then a postponement of the peak effect of advertising lowers the long-run equilibrium rate of advertising* (§2.3).

PROPOSITION 3. *If the diffusion of advertising is propagated by word-of-mouth (system dynamics* (3.2)), *then for both, linear and nonlinear advertising effectiveness, multiple equilibria occur. This means that the advertising policy depends critically on the initial sales level* (§3.1).

PROPOSITION 4. *In a large class of cumulative sales models, the optimal advertising rate is increasing if advertising becomes more efficient during the product life cycle* (§4).

PROPOSITION 5. *In marketing models with different types of advertising, the optimal advertising rate need not decrease monotonically over time. In particular, awareness advertising can decline, while trial advertising is expected to increase* (§5.1).

PROPOSITION 6. *When there is an additional effect on sales if goodwill advertising exceeds its long-run average, then pulsing advertising can be optimal* (§5.2).

PROPOSITION 7. *All other factors remaining constant, the larger the interaction effect of manufacturer's goodwill on the channel sales, the greater is the possibility of channel coordination* (§7.4); *see Chintagunta and Jain* (1992a).[2]

[2] This research was supported in part by Grant A4619 from NSERC of Canada and by Grant P6601 from the Austrian Science Foundation. Comments and suggestions from E. J. Dockner, J. Eliashberg, S. Jørgensen, A. Luhmer, A. Mitchell, J. Ringbeck, G. Sorger, and K. Spremann are gratefully acknowledged.

References

Abad, P. L., "An Optimal Control Approach to Marketing-Production Planning," *Optimal Control Applications and Methods*, 3, 1 (January–March 1982a), 1–13.

——, "Approach to Decentralized Marketing-Production Planning," *International J. Systems Sci.*, 13, 3 (1982b), 227–235.

——, "A Hierarchical Optimal Control Model for Coordination of Functional Decisions in a Firm," *European J. Operational Res.*, 32 (1987), 62–75.

——, "Multi-Product Multi-Market Model for Coordination of Marketing-Production Decisions," *International J. Systems Sci.*, 20 (1989), 2011–2027.

Assmus, G., J. U. Farley and D. R. Lehmann, "How Advertising Affects

Sales: Meta Analysis of Econometric Results," *J. Marketing Res.*, (February 1984), 65–74.

Basar, T. and G. J. Olsder, *Dynamic Noncooperative Game Theory*, Academic Press, London, 1982.

Bass, F. M., "A New Product Growth Model for Consumer Durables," *Management Sci.*, 15, 5 (January 1969), 215–227.

——, "The Relationship Between Diffusion Rates, Experience Curves and Demand Elasticities for Consumer Durable Technological Innovations," *J. Business*, 53, H2 (1980), S51–S67.

—— and D. G. Clarke, "Testing Distributed Lag Models of Advertising Effect," *J. Marketing Res.*, 9 (August 1972), 298–308.

—— and R. P. Leone, "Empirical Estimation of Bimonthly Relations From Annual Data," *Management Sci.*, 29, 1 (1983), 1–11.

Bensoussan, A., A. Bultez and P. Naert, "Leader's Dynamic Marketing Behavior in Oligopoly," in A. Bensoussan et al., Eds., *TIMS Studies in the Management Sciences*, Vol. 9, North-Holland, Amsterdam, 1978, 123–145.

Blattberg, R. C. and A. P. Jeuland, "Micromodeling Approach to Investigate the Advertising-Sales Relationships," *Management Sci.*, 27, 9 (September 1981), 988–1005.

Boston Consulting Group, *Perspectives on Experience*, The Boston Consulting Group, Boston, MA, 1972.

Bourguignon, F. and S. P. Sethi, "Dynamic Optimal Pricing and (Possibly) Advertising in the Face of Various Kinds of Potential Entrants," *J. Econ. Dynamics and Control*, 3 (1981), 119–140.

Bultez, A. V. and Ph. A. Naert, "Does Lag Structure Really Matter in Optimizing Advertising Expenditures?", *Management Sci.*, 25, 5 (May 1979), 454–465.

—— and ——, "When Does Lag Structure Really Matter . . . Indeed?", *Management Sci.*, 34, 7 (1988), 909–916.

Caines, P., S. Sethi and T. Brotherton, "Impulse Response Identification and Causality Detection for the Lydia Pinkham Data," *Annals of Economic and Social Measurement*, 6, 2 (1977), 147–163.

Carpenter, G. S., L. G. Cooper, D. M. Hanssens and D. F. Midgley, "Modeling Asymmetric Competition," *Marketing Sci.*, 7 (Fall 1988), 393–412.

Case, J. H., *Economics and the Competitive Process*, New York University, New York, 1979.

Chatterjee, R. and J. Eliashberg, "The Innovation Diffusion Process in a Heterogeneous Population: A Micromodeling Approach," *Management Sci.*, 36, 9 (1990), 1057–1079.

Chintagunta, P. and D. Jain, "A Study of Manufacturer-Retailer Marketing Strategies: A Differential Game Approach," *Lecture Notes in Control and Information Sciences*, Springer, New York, 1994, to appear.

—— and ——, "A Dynamic Model of Channel Member Strategies for Marketing Expenditures," *Marketing Sci.*, 11, 2 (1992a), 168–188.

—— and ——, "Dynamic Duopoly Models of Advertising Competition: Estimation and a Specification Test," Working Paper, Northwestern University, Evanston, IL, 1992b.

—— and N. J. Vilcassim, "An Empirical Investigation of Advertising Strategies in a Dynamic Duopoly," *Management Sci.*, 38, 9 (1992), 1230–1244.

—— and ——, "Marketing Investment Decisions in a Dynamic Duopoly: A Model and Empirical Analysis," Working Paper, Cornell University, Ithaca, NY and Northwestern University, Evanston, IL, 1991.

Conrad, K., "Advertising, Quality·and Informationally Consistent Prices," *Zeitschrift für gesamte Staatswissenschaft*, 138, 4 (December 1982), 680–694.

——, "Quality, Advertising and the Formation of Goodwill under Dynamic Conditions," in G. Feichtinger, Ed., *Optimal Control Theory and Economic Analysis*, Vol. 2, North-Holland, Amsterdam, 1985, 215–234.

Deal, K. R., "Optimizing Advertising Expenditures in a Dynamic Duopoly," *Operations Research*, 27, 4 (July–August 1979), 682–692.

——, S. P. Sethi and G. L. Thompson, "A Bilinear-Quadratic Differential Game in Advertising," in P.-T. Liu and J. G. Sutinen, Eds., *Control Theory in Mathematical Economics*, Dekker, New York, 1979, 91–109.

Dehez, P. and A. Jacquemin, "A Note on Advertising Policy under Uncertainty and Dynamic Conditions," *Journal of Industrial Economics*, 24 (1975), 73–78.

Dockner, E. and S. Jorgensen, "Optimal Advertising Policies for Diffusion Models of New Product Innovation in Monopolistic Situations," *Management Sci.*, 34, 1 (1988), 119–130.

——, G. Feichtinger and G. Sorger, "Interaction of Pricing and Advertising Under Dynamic Conditions," DGOR Operations Research Proceedings, Springer, Berlin, 1989, 547–554.

—— and ——, "Dynamic Advertising and Pricing in an Oligopoly: A Nash Equilibrium Approach," in T. Basar, Ed., *Dynamic Games and Applications in Economics*, Springer, Berlin, 1986, 238–251.

—— and H. Takahashi, "Advertising and Dynamic Oligopolistic Competition: A Note," Working Paper, University of Saskatchewan, Saskatoon, Canada, 1988.

—— and S. Jorgensen, "New Product Advertising in Dynamic Oligopolies," *Zeitschrift für Operations Research*, 36, 5 (1992), 459–473.

Dodson, J. A. and E. Muller, "Models for New Product Diffusion Through Advertising and Word-of-Mouth," *Management Sci.*, 24, 15 (1978), 1568–1578.

Dolan, R. J. and A. P. Jeuland, "Experience Curves and Dynamic Demand Models: Implications for Optimal Pricing Strategies," *J. Marketing*, 45 (Winter 1981), 52–62.

——, —— and E. Muller, "Models of New Product Diffusion: Extension to Competition Against Existing and Potential Firms Over Time," in V. Mahajan and J. Wind, Eds., *Innovation Diffusion Models of New Product Acceptance*, Ballinger, Cambridge, MA, 1986, 117–150.

Dorfman, R. and P. O. Steiner, "Optimal Advertising and Optimal Quality," *American Economic Review*, 44 (1954), 826–836.

Duffie, D., *Security Markets: Stochastic Models*, Academic Press, Boston, MA, 1988.

Ekman, E., "A Dynamic Advertising Model of a Labor-Managed Firm," presented at European Institute of Advanced Studies in Management, Brussels, October 30, 1980.

Eliashberg, J. and R. Chatterjee, "Analytical Models of Competition with Implications for Marketing: Issues, Findings, and Outlook," *J. Marketing Research*, 22 (August 1985), 237–261.

—— and G. L. Lilien (Eds.), *Marketing*, Handbooks in O.R. and M.S., Vol. 5, North-Holland, Amsterdam, 1993.

—— and R. Steinberg, "Marketing-Production Decisions in an Industrial Channel of Distribution," *Management Sci.*, 33, 8 (1987), 981–1000.

Erickson, G. M., "Optimal Price-Advertising Interaction for New Consumer Durables," Working Paper, University of Washington, 1982.

——, "A Model of Advertising Competition," *J. Marketing Research*, 22 (August 1985), 297–304.

——, *Dynamic Models of Advertising Competition*, Kluwer Academic Publishers, Boston, MA, 1991.

——, "Empirical Analysis of Closed-Loop Duopoly Advertising Strategies," *Management Sci.*, 38 (1992), 1732–1749.

Feichtinger, G., "Optimal Pricing in a Diffusion Model with Concave Price-Dependent Market Potential," *O. R. Letters*, 1, 6 (1982a), 236–240.

——, "Saddle Point Analysis in a Price-Advertising Model," *J. Econ. Dynamics and Control*, 4 (1982b), 319–340.

——, "The Nash Solution of an Advertising Game: Generalization of a Model by Leitmann and Schmitendorf," *IEEE Trans. Automatic Control*, AC-28, 11 (1983), 1044–1048.

—— and E. Dockner, "A Note to Jorgensen's Logarithmic Advertising Differential Game," *Zeitschrift für Operations Research*, 28 (1984), B133–B153.

—— and R. F. Hartl, *Optimale Kontrolle Ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, deGruyter, Berlin, 1986.

—— and S. Jorgensen, "Differential Game Models in Management Science," *European J. Operational Res.*, 14 (1983), 137–155.

——, A. Luhmer and G. Sorger, "Optimal Price and Advertising Policy for a Convenience Goods Retailer," *Marketing Sci.*, 7, 2 (1988), 187–201.

Feinberg, F. M., "Pulsing Policies for Aggregate Advertising Models," *Marketing Sci.*, 11, 3 (Summer 1992), 221–234

Ferreyra, G., "The Optimal Control Problem for the Vidale-Wolfe Advertising Model Revisited," *Optimal Control Applications and Methods*, 11 (1990), 363–368.

Fershtman, C., "Goodwill and Market Shares in Oligopoly," *Economica*, 51 (1984), 271–281.

——, V. Mahajan and E. Muller, "Market Share Pioneering Advantage: A Theoretical Approach," *Management Sci.*, 36, 8 (1990), 900–918.

Fine, C. H., "Quality Improvement and Learning in Productive Systems," *Management Sci.*, 32, 10 (1986), 1301–1315.

Fleming, W. H. and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

——, S. P. Sethi and H. M. Soner, "An Optimal Stochastic Production Planning Problem with Randomly Fluctuating Demand," *SIAM J. Control Optim.*, 25, 6 (1987), 1494–1502.

Friedman, J. W., "Advertising and Oligopolistic Equilibrium," *Bell J. Econ.*, 14, 2 (1983), 464–473.

Funke, U. H., *Mathematical Models in Marketing: A Collection of Abstracts*, Lecture Notes in Econ. and Math. Systems, Vol. 132, Springer, Berlin, Heidelberg, New York, 1976.

Gasmi, F. and Q. H. Vuong, "An Econometric Analysis of Some Duopolistic Games in Prices and Advertising," Working Paper, University of Southern California, Los Angeles, CA, 1988.

Gaugusch, J., "The Noncooperative Solution of a Differential Game: Advertising Versus Pricing," *Optimal Control Applications and Methods*, 5, 4 (1984), 353–360.

Glaister, S., "Advertising Policy and Returns to Scale in Markets Where Information Is Passed Between Individuals," *Economica*, 41 (May 1974), 139–156.

Gould, J. P., "Diffusion Processes and Optimal Advertising Policy," in E. S. Phelps, et al. (Eds.), *Microeconomic Foundation of Employment and Inflation Theory*, Norton, New York, 1970, 338–368.

Gromball, P. von and C. Becker, "Bilineare Systeme zur Bestimmung Optimaler Werbestrategien," *Zeitschrift für Operations Research-A*, 23, 3 (1979), 117–126.

Guckenheimer, J. and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.

Hahn, M. and J. S. Hyun, "Advertising Cost Interpretations and the Optimality of Pulsing," *Management Sci.*, 37, 2 (February 1991), 157–169.

Hanssens, D. M., "Market Response, Competitive Behavior and Time Series Analysis," *J. Marketing Res.*, 17 (November 1980), 470–485.

——, L. J. Parsons and R. L. Schultz, *Market Response Models: Econometric and Time Series Analysis*, Kluwer Academic Publishers, Norwell, MA, 1990.

Hartl, R. F., "Optimal Control of Non-Linear Advertising Models with Replenishable Budget," *Optimal Control Applications and Methods*, 3, 1 (January 1982), 53–65.

——, "Optimal Dynamic Advertising Policies for Hereditary Processes," *J. Optimization Theory and Applications*, 43, 1 (May 1984), 51–72.

—— and G. Feichtinger, "A New Sufficient Condition for Most Rapid Approach Paths," *J. Optimization Theory and Applications*, 54, 2 (August 1987), 403–411.

——, S. P. Sethi and R. Vickson, "A Survey of the Maximum Principle for Optimal Control Problems with State Constraints," *Research Report* Nr. 153, (March), Institute for Econometrics, Operations Research, and Systems Theory, Vienna University of Technology, Vienna, Austria, 1992.

Helson, D., *Adaptation Level Theory*, Harper and Row, New York and Evanston, 1964.

Hjorth-Andersen, C., "The Concept of Quality and the Efficiency of Markets for Consumer Products," *J. Consumer Res.*, 11 (September 1984), 708–718.

Holt, C. C., F. Modigliani, J. F. Muth and H. A. Simon, *Planning*

Production, Inventories, and Workforce, Prentice-Hall, Englewood Cliffs, NJ, 1960.

Horsky, D., "An Empirical Analysis of the Optimal Advertising Policy," *Management Sci.*, 23, 10 (June 1977a), 1037–1049.

——, "Market Share Response to Advertising: An Example of Theory Testing," *J. Marketing Res.*, 14 (February 1977b), 10–21.

——, "Diffusion Models Incorporating Product Benefits, Price, Income, and Information," *Marketing Sci.*, 9 (Fall 1990), 342–365.

—— and K. Mate, "Dynamic Advertising Strategies of Competing Durable Good Producers," *Marketing Sci.*, 7, 4 (Fall 1988), 356–367.

—— and L. S. Simon, "Advertising and the Diffusion of New Products," *Marketing Sci.*, 2, 1 (Winter 1983), 1–17.

Ireland, N. J. and H. G. Jones, "Optimality in Advertising: A Control Theory Approach," *IEE Conf. Pub. No.* 101, Institution of Electrical Engineers, Hitchin, Herts., England, 1973.

Jacqnemin, A. P., "Optimal Control and Advertising Policy," *Metroeconomica*, 25 (1973), 200–209.

—— and J. Thisse, "Strategy of the Firm and Market Structure: An Application of Optimal Control Theory," in K. Cowling (Ed.), *Market Structure and Corporate Behaviour*, Gray-Mills, London, 1972.

Jedidi, K., J. Eliashberg and W. DeSarbo, "Optimal Advertising and Pricing for a Three-Stage Time-Lagged Monopolistic Diffusion Model Incorporating Income," *Optimal Control Applications and Methods*, 10 (1989), 313–331.

Jeuland, A. P. and R. J. Dolan, "An Aspect of New Product Planning: Dynamic Pricing," in A. A. Zoltners (Ed.), *Marketing Planning Models*, TIMS Studies in the Management Science, Vol. 18, North-Holland, Amsterdam, 1982, 1–21.

Jones, P., "Analysis of a Dynamic Duopoly Model of Advertising," *Math. Oper. Res.*, 8, 1 (February 1983), 122–134.

Jorgensen, S., "A Survey of Some Differential Games in Advertising," *J. Econ. Dynamics and Control*, 4 (1982a), 341–369.

——, "A Differential Games Solution to a Logarithmic Advertising Model," *J. Oper. Res. Soc.*, 33, 5 (May 1982b), 425–432.

——, "Optimal Control of a Diffusion Model of New Product Acceptance with Price-Dependent Total Market Potential," *Optimal Control Applications and Methods*, 4, 3 (July 1983), 269–276.

——, "Optimal Cooperative Advertising in a Dynamic Vertical Marketing System: A Differential Games Approach," Working Paper, Copenhagen School of Economics and Business Administration, Institute of Theoretical Statistics, June, 1984.

Journal of Marketing Research, Special Issue on Competition in Marketing, 22, 3, 1985.

Kalish, S., "Monopolist Pricing with Dynamic Demand and Production Cost," *Marketing Sci.*, 2, 2 (Spring 1983), 135–159.

——, "A New Product Adoption Model with Price, Advertising, and Uncertainty," *Management Sci.*, 31, 12 (December 1985), 1569–1585.

—— and G. L. Lilien, "Optimal Price Subsidy Policy for Accelerating the Diffusion of Innovation," *Marketing Sci.*, 2, 4 (Fall 1983), 407–420.

—— and S. K. Sen, "Diffusion Models and the Marketing Mix for Single Products," in V. Mahajan and J. Wind (Eds.), *Innovation Diffusion Models of New Product Acceptance*, Ballinger, Cambridge, MA, 1986, 87–116.

—— and ——, "Diffusion Models and the Marketing Mix for Single Products," in V. Mahajan and Y. Wind (Eds.), *Innovation Diffusion Models of New Product Acceptance*, Ballinger, Cambridge, MA, 1986, 87–116.

Kamien, M. I. and N. L. Schwartz, "Optimal Maintenance and Sale Age for a Machine Subject to Failure," *Management Sci.*, 17 (1971), 427–449.

Karatzas, I., J. P. Lehoczky, S. P. Sethi and S. Shreve, "Explicit Solution of a General Consumption/Investment Problem," *Math. Oper. Res.*, 11, 2 (1986), 261–294.

—— and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 1988.

Kotowitz, Y. and F. Mathewson, "Advertising, Consumer Information, and Product Quality," *Bell J. Economics*, 10, 2 (Autumn 1979a), 566–588.

—— and ——, "Informative Advertising and Welfare," *American Econ. Review*, 128 (June 1979b), 284–294.

Kouvelis, P. and S. Mukhopadhyay, "Competing on Quality: A Strategic Planning Approach for Product Quality with the Use of a Control Theoretical Model," Working Paper, University of Texas at Austin and University of Wisconsin, July 1990.

Lambin, J.-J., P. A. Naert and A. Bultez, "Optimal Marketing Behavior in an Oligopoly," *European Econ. Review*, 6 (1975), 105–128.

Lambin, J. J., *Advertising, Competition and Market Conduct in Oligopoly Over Time*, North-Holland, Amsterdam, 1976.

Lavidge, R. J. and G. A. Steiner, "A Model for Predictive Measurements of Advertising Effectiveness," *J. Marketing*, 25, 6 (1961), 59–62.

Leitmann, G. and W. E. Schmitendorf, "Profit Maximization Through Advertising: A Nonzero Sum Differential Game Approach," *IEEE Trans. on Automatic Control* AC-23, 4 (August 1978), 645–650.

Lignell, J. and M. P. T. Tuominen, "An Advertising Control Model of Two State Variables," *European J. Oper. Res.*, 24, 1 (1983), 77–84.

Little, J. D. C., "Aggregate Advertising Models: The State of the Art," *Oper. Res.*, 27, 4 (July/August 1979), 629–667.

——, "Comment on 'Advertising Pulsing Policies . . .', by V. Mahajan and E. Muller," *Marketing Sci.*, 5, 2 (Spring 1986), 107–108.

Luhmer, A., A. Steindl, G. Feichtinger, R. F. Hartl and G. Sorger, "ADPULS in Continuous Time," *European J. Oper. Res.*, 34, 2 (March 1988), 171–177.

Luptacik, M., "Optimal Price and Advertising Policy Under Atomistic Competition," *J. Econ. Dynamics and Control*, 4 (1982), 57–71.

Magat, W. A., J. M. McCann and R. C. Morey, "When Does Lag Structure Really Matter in Optimizing Advertising Expenditures?", *Management Sci.*, 32, 2 (February 1986), 182–193.

——, —— and ——, "Reply to When Does Lag Structure Really Matter ... Indeed?", *Management Sci.*, 34, 7 (July 1988), 917–918.

Mahajan, V. and E. Muller, "Innovation Diffusion and New Product

Growth Models in Marketing," *J. Marketing*, 43 (Fall 1979), 55–68.

Mahajan, V. and E. Muller, "Advertising Pulsing Policies for Generating Awareness for New Products," *Marketing Sci.*, 5, 2 (Spring 1986), 89–111.

—— and R. A. Peterson, *Models for Innovation Diffusion*, Sage, Beverly Hills, 1985.

—— and Y. Wind, Eds., *Innovation Diffusion Models of New Product Acceptance*, Ballinger, Cambridge, MA, 1986.

——, E. Muller and R. A. Kerin, "Introduction Strategy for New Products with Positive and Negative Word-of-Mouth, *Management Sci.*, 30, 12 (December 1984), 1389–1404.

——, S. Sharma and J. Wind, "An Approach to Repeat-Purchase Diffusion Analysis," *Proc. American Marketing Educators Conf.*, American Marketing Association, Chicago, IL, 1983, 442–446.

Marketing Science, Special Issue on Competitive Marketing Strategy, 7, 4 (1988).

Mehlmann, A., *Applied Differential Games*, Plenum Press, New York, 1988.

Mesak, H. I., "On Modeling Advertising Pulsing Decisions," *Decision Sciences*, 16 (1985), 25–42.

—— and A. F. Darrat, "On Comparing Alternative Advertising Policies for Pulsation," *Decision Sciences*, 23 (1992), 541–564.

Midgley, D. F., "A Simple Mathematical Theory of Innovative Behavior," *J. Consumer Res.*, 3 (1976), 31–41.

Monahan, G. E., "A Pure Birth Model of Optimal Advertising with Word-of-Mouth," *Marketing Sci.*, 3, 2 (Spring 1984), 169–178.

Montgomery, D. B. and A. J. Silk, "Estimating Dynamic Effects of Market Communications Expenditures," *Management Sci.*, 18, 10 (June 1972), B485–B501.

Morton, E., A. Mitchell and E. Zemel, "A Discrete Maximum Principle Approach to a General Dynamic Market Response Model," *TIMS Studies in the Management Sciences*, Vol. 18, North-Holland, Amsterdam, 1982, 117–140.

Muller, E., "Trial / Awareness Advertising Decision: A Control Problem with Phase Diagrams with Non-Stationary Boundaries," *J. Econ. Dynamics and Control*, 6 (1983), 333–350.

Näslund, B., "Consumer Behaviour and Optimal Advertising," *J. Oper. Res. Soc.*, 30, 3 (1979), 237–243.

Nelson, P., "Advertising as Information," *J. Political Econ.*, 81, 4 (July / August 1974), 729–754.

——, "Information and Consumer Behavior," *J. Political Econ.*, 78, 2 (March / April 1970), 311–329.

Nerlove, M. and K. J. Arrow, "Optimal Advertising Policy under Dynamic Conditions," *Economica*, 29 (1962), 129–142.

Nicosia, F. M., *Consumer Decision Processes*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

Olsder, G. J., "Some Thoughts About Simple Advertising Models as Differential Games and the Structure of Coalitions," in Y. C. Ho and S. K. Mitter (Eds.), *Directions in Large Scale Systems*, Plenum, New York, 1976, 187–205.

Opitz, O. and K. Spremann, "Optimale Steuerung von Kaufverhaltensprozessen," in E. Topritzhofer (Ed.), *Marketing: Neuere Ergebnisse aus Forschung und Praxis*, Gabler, Wiesbaden, 1978, 285–324.

Ozga, S., "Imperfect Markets Through Lack of Knowledge," *Quart. J. Econ.*, 74 (1960), 29–52.

Palda, K. S., *The Measurement of Cumulative Advertising Effects*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

Park, S. and M. Hahn, "Pulsing in a Discrete Model of Advertising Competition," *J. Marketing Res.*, 28 (November 1991), 397–405.

Parlar, M., "A Problem in Jointly Optimal Production and Advertising Decisions," *International J. Systems Sci.*, 17, 9 (1986), 1373–1380.

Pauwels, W., "Optimal Dynamic Advertising Policies in the Presence of Continuously Distributed Time Lags," *J. Optimization Theory and Applications*, 22, 1 (May 1977), 79–89.

Pekelman, D. and E. Tse, "The Copy Life Cycle Decisions: Timing of a Copy Replacement, Investment in a New Copy, and Advertising Expenditures During the Cycle," *Proc. American Marketing Educators Conf.*, Memphis, TN, AMA, Chicago, IL, 1976, 67–70.

—— and ——, "Experimentation and Budgeting in Advertising: An Adaptive Control Approach," *Oper. Res.*, 28, 2 (March–April 1980), 321–347.

—— and G. Rausser, "Adaptive Control: Survey of Methods and Applications," in Bensoussan et al. (Eds.), *Applied Optimal Control*, *TIMS Studies in the Management Sciences*, Vol. 9, North-Holland, Amsterdam, 1978, 89–120.

—— and S. Sethi, "Advertising Budgeting, Wearout and Copy Replacement," *J. Oper. Res. Soc.*, 29, 7 (1978), 651–659.

Picconi, M. J. and C. L. Olson, "Advertising Decision Rules in a Multibrand Environment: Optimal Control Theory and Evidence," *J. Marketing Res.*, 15 (1978), 82–92.

Raman, K., "Dynamic Optimization of the Marketing Mix in a Stochastic Environment," Working Paper, University of Florida, Gainesville, FL, 1988.

——, "Stochastically Optimal Advertising Policies Under Dynamic Conditions: The Ratio Rule," *Optimal Control Applications and Methods*, 11 (1990), 283–288.

——, "Feedback Advertising Strategies in an Oligopoly Under Uncertainty," Working Paper, University of Florida, Gainesville, FL, 1991.

Rao, A. G., *Quantitative Theories in Advertising*, Wiley, New York, 1970.

Rao, R. C., "Advertising Decisions in Oligopoly: An Industry Equilibrium Analysis," *Optimal Control Applications and Methods*, 5, 4 (October–December 1984), 331–344.

——, "A Note on Optimal and Near Optimal Price and Advertising Strategies," *Management Sci.*, 31, 3 (March 1985), 376–377.

——, "Estimating Continuous Time Advertising-Sales Models," *Marketing Sci.*, 5, 2 (Spring 1986), 125–142.

——, "Impact of Competition on Strategic Marketing Decisions," in *Interface of Marketing and Strategy*, G. Day, B. Weitz, and R. Wensley (Eds.), JAI Press, Greenwich, CT, 1990.

Ringbeck, J., "Mixed Quality and Advertising Strategies Under Asymmetric Information," in G. Feichtinger (Ed.), *Optimal Control Theory and Economic Analysis*, Vol. 2, North-Holland, Amsterdam, 1985, 197–214.

——, *Qualitäts- und Werbestrategien bei Qualitätsunsicherheit der Konsumenten*, Beitrage Betriebswirtschaftlichen Forschung, Vol. 60, Gabler, Wiesbaden, 1986a.

Ringbeck, J., "Werbeplanung fur Low-Involvement Produkete," *Marketing ZfP*, 4 (November 1987).

——, "Optimal Advertising Strategies under Dynamic Conditions—A Generalization of Some Well-Known Advertising Models," Working Paper 8616, Universität Osnabrück, 1986b.

Rishel, R., "A Partially Observed Advertising Model," in G. Feichtinger, Ed., *Optimal Control Theory and Economic Analysis*, Vol. 2, North-Holland, Amsterdam, 1985, 253–262.

Robinson, B. and C. Lakhani, "Dynamic Price Models for New-Product Planning," *Management Sci.*, 21, 10 (June 1975), 1113–1122.

Roberts, M. J. and L. Samuelson, "An Empirical Analysis of Dynamic Non-Price Competition in an Oligopolistic Industry," *RAND J. Econ.*, 19 (1988), 200–220.

Sappington, D. E. M. and B. Wernerfelt, "To Brand or Not To Brand? A Theoretical and Empirical Question," *J. Business*, 58, 3 (1985), 279–294.

Sasieni, M. W., "Optimal Advertising Expenditure," *Management Sci.*, 18, 4 (December 1971), P 64–P 72.

Schmalensee, R., *The Economics of Advertising*, North-Holland, Amsterdam, 1972.

——, "Brand Loyalty and Barriers to Entry," *Southern Econ. J.*, 40 (1974), 579–588.

——, "A Model of Advertising and Product Quality," *J. Political Econ.*, 86, 3 (1978), 485–487.

Seidman, T. I., S. P. Sethi and N. A. Derzko, "Dynamics and Optimization of a Sales-Advertising Model," *J. Optimization Theory and Applications*, 52, 3 (1987), 443–462.

Sethi, S. Prakash, "A Comparison Between the Effect of Pulse Versus Continuous Television Advertising on Buyer Behaviour," *Proc. American Marketing Assoc.*, American Marketing Association, Chicago, IL, 1971, 563–567.

Sethi, S. Pal, "Dynamic Optimal Control Models in Advertising: A Survey," *SIAM Review*, 19, 4 (October 1977a), 685–725.

——, "Nearest Feasible Paths in Optimal Control Problems: Theory, Examples, and Counterexamples," *J. Optimization Theory and Applications*, 23, 4 (December 1977b), 563–579.

——, "Optimal Advertising for the Nerlove-Arrow Model Under a Budget Constraint," *Oper. Res. Quarterly*, 28, 3 (1977c), 683–693.

——, "Optimal Advertising Policy with the Contagion Model," *J. Optimization Theory and Applications*, 29, 4 (December 1979a), 615–627.

——, "A Note on the Nerlove-Arrow Model under Uncertainty," *Oper. Res.*, 27, 4 (July–August 1979b), 839–842.

——, "Deterministic and Stochastic Optimization of a Dynamic Advertising Model," *Optimal Control Applications and Methods*, 4, 2 (June 1983a), 179–184.

——, "Optimal Long-Run Equilibrium Advertising Level for the Blattberg-Jeuland Model," *Management Sci.*, 29, 12 (December 1983b), 1436–1443.

—— and G. L. Thompson, *Optimal Control Theory: Applications to Management Science*, M. Nijhoff, Boston, 1981a.

——, "A Tutorial on Optimal Control Theory," *INFOR*, 19, 4 (Nov. 1981b), 279–291.

—— and S. C. Lee, "Optimal Advertising for the Nerlove-Arrow Model Under a Replenishable Budget," *Optimal Control Applications and Methods*, 2, 2 (April–June 1981), 165–173.

—— and Q. Zhang, "Multilevel Hierarchical Controls in Dynamic Stochastic Marketing-Production System," *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, AZ, Dec. 16–18, 1992, 2090–2095.

—— and ——, "Multilevel Hierarchical Decision Making in Stochastic Marketing-Production Systems," Working Paper, University of Toronto, Toronto, Ontario, Canada, 1993; *SIAM J. Control & Optim.* (to appear).

Shapiro, C. "Consumer Information, Product Quality, and Seller Reputation," *Bell J. Econ.*, 13, 1 (1982), 20–35.

Simon, H., "ADPULS: An Advertising Model with Wear Out and Pulsation," *J. Marketing Res.*, 19 (August 1982), 352–363.

——, *Price Management*, Elsevier, New York, 1989.

—— and K.-H. Sebastian, "Diffusion and Advertising: The German Telephone Campaign," *Management Sci.*, 33, 4 (April 1987), 451–466.

—— and Q. Zhang, "Multilevel Hierarchical Controls in Dynamic Stochastic Marketing-Production System," *Proc. 31st IEEE Conf. on Decision and Control*, Tucson, AZ, Dec. 16–18, 1992, 2090–2095.

—— and ——, "Multilevel Hierarchical Decision Making in Stochastic Marketing-Production Systems," Working Paper, University of Toronto, Toronto, Ontario, Canada, 1993.

Shapiro, "Consumer Information, Product Quality, and Seller Reputation," *Bell J. Econ.*, 13, 1 (1982), 20–35.

Simon, H., "ADPULS: An Advertising Model with Wear Out and Pulsation," *J. Marketing Res.*, 19 (August 1982), 352–363.

—— and E. Kucher, *Price Management*, Elsevier, New York, 1989.

—— and K.-H. Sebastian, "Diffusion and Advertising: The German Telephone Campaign," *Management Sci.*, 33, 4 (April 1987), 451–466.

Singh, M. G., *Dynamical Hierarchical Control*, North-Holland, Amsterdam, 1980.

Sorger, G., "Optimal Planning of Work Force, Capital, Marketing, and Production by Hierarchical Control Theory," Forschungsbericht Nr. 101, Institut für Ökonometrie und Operations Research, Technische Universität Wien, Austria, 1986.

——, "Das Preisbewußtsein der Konsumenten und sein Einfluß auf die Optimalität zyklischer Marketingstrategien," *Quartalshefte der Girozentrale*, 22, 4 (1987), 119–125.

——, "Competitive Dynamic Advertising: A Modification of the Case Game," *J. Econ. Dynamics and Control*, 13 (1989), 55–80.

Spence, M. and D. Starrett, "Most Rapid Approach Paths in Accumulation Problems," *International Econ. Review*, 16 (1975), 388–403.

Spremann, K., "Optimale Preispolitik bei Dynamischen Deterministischen Absatzmodellen," *Zeitschrift für Nationalökonomie*, 35, 1–2 (1975), 63–76.

Spremann, K., "Hybrid Product Life Cycles and the Nerlove-Arrow Model," Working Paper, Universität Ulm, July 1981.

——, "The Signaling of Quality by Reputation," in G. Feichtinger, Ed., *Optimal Control Theory and Econ. Analysis*, Vol. 2, North-Holland, Amsterdam, 1985, 235–252.

Stigler, G., "The Economics of Information," *J. Political Econ.*, 69 (1961), 213–225.

Sultan, F., J. U. Farley and D. R. Lehmann, "A Meta-Analysis of Application of Diffusion Models," *J. Marketing Res.*, 27 (February 1990), 70–77.

Sweeney, D. J., P. L. Abad and R. J. Dornoff, "Finding an Optimal Dynamic Advertising Policy," *International J. Systems Science*, 5, 10 (1974), 987–994.

Tapiero, C. S., "A Stochastic Model of Sales Response to Advertising," *Metroeconomica*, 29 (1977), 159–167.

——, "Optimum Advertising and Goodwill under Uncertainty," *Oper. Res.*, 26, 3 (May–June 1978), 450–463.

——, "A Generalization of the Nerlove-Arrow Model to Multi-Firms Advertising under Uncertainty," *Management Sci.*, 25, 9 (September 1979), 907–915.

——, "Optimal Product Quality and Advertising," *INFOR*, 19, 4 (November 1981), 311–318.

——, "Optimal Control of a Stochastic Model of Advertising," in G. Feichtinger (Ed.), *Optimal Control Theory and Econ. Analysis*, North-Holland, Amsterdam, 1982a, 287–300.

——, "A Stochastic Model of Consumer Behavior and Optimal Advertising," *Management Sci.*, 28, 9 (September 1982b), 1054–1064.

——, "Stochastic Diffusion Models with Advertising and Word-of-Mouth Effects," *European J. Oper. Res.*, 12, 4 (1983), 348–356.

——, *Applied Stochastic Models and Control in Management*, North-Holland, Amsterdam, 1988.

—— and J. U. Farley, "Optimal Control of Sales Force Effort in Time," *Management Sci.*, 21, 9 (May 1975), 976–985.

—— and ——, "Using an Uncertainty Model to Assess Sales Response to Advertising," *Decision Sci.*, 12 (1981), 441–455.

——, J. Eliashberg and Y. Wind, "Risk Behaviour and Optimum Advertising with a Stochastic Dynamic Sales Response," *Optimal Control Applications and Methods*, 8, 3 (July–September 1987), 299–304.

Teng, J.-T. and G. L. Thompson, "Oligopoly Models for Optimal Advertising when Production Costs Obey a Learning Curve," *Management Sci.*, 29, 9 (September 1983), 1087–1101.

—— and ——, "Optimal Strategies for General Price-Advertising Models," in G. Feichtinger (Ed.), *Optimal Control Theory and Economic Analysis*, Vol. 2, North-Holland, Amsterdam, 1985, 183–195.

Thepot, J., "Marketing and Investment Policies of Duopolists in a Growing Industry," *J. Econ. Dynamics and Control*, 5 (1983), 387–404.

Thompson, G. L., "An Optimal Control Model of Advertising Pulsation and Wearout," in J. W. Keon, Ed., *ORSA/TIMS Special Interest Conf. on Market Measurement and Analysis*, New York, 1981, 34–43.

——, S. P. Sethi and J.-T. Teng, "Strong Planning and Forecast Horizons for a Model with Simultaneous Price and Production Decisions, *European J. Oper. Res.*, 16 (1984), 378–388.

—— and J.-T. Teng, "Optimal Pricing and Advertising Policies for New Product Oligopoly Models," *Marketing Sci.*, 3, 2 (Spring 1984), 148–168.

Tse, E., Y. Bar-Shalom and L. Meier, "Wide-Sense Adaptive Dual Control of Stochastic Nonlinear Systems," *IEEE Trans. Automatic Control*, AC-18 (1973), 98–108.

Tsurumi, H. and Y. Tsurumi, "An Econometric Model of a Japanese Pharmaceutical Company," in K. Palda (Ed.), *Reading in Managerial Economics*, Prentice-Hall, Englewood Cliffs, NJ, 1973, 290–316.

Urban, G. L. and J. R. Hauser, *Design and Marketing of New Products*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

Verma, V. K., "A Price Theoretic Approach to the Specification and Estimation of the Sales-Advertising Function," *J. Business*, 53, 3 (1980), 115–137.

Vidale, M. L. and H. B. Wolfe, "An Operations Research Study of Sales Response to Advertising," *Oper. Res.*, 5, 3 (June 1957), 370–381.

Villas-Boas, J. M., "Predicting Advertising Policies in an Oligopoly: A Model and Empirical Test," *Marketing Sci.*, 12 (Winter 1993), 88–102.

Ward, R. W., "Measuring Advertising Decay," *J. Advertising Res.*, 16 (August 1976), 37–41.

Welam, U. P., "Optimal and Near Optimal Price and Advertising Strategies for Finite and Infinite Horizons," *Management Sci.*, 28, 11 (December 1982), 1313–1327.

Yang, J. H., B. H. Jeong and S. H. Kim, "A Dynamic System Model for Control of Advertising Expenditure," Working Paper, Korea Advanced Institute of Science and Technology, Seoul, Korea, 1987.

Accepted by Jehoshua Eliashberg; received February 13, 1990. This paper has been with the authors 24 months for 2 revisions.